Transport kontenerowy,
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SEMI-MARKOV APPROACH APLIED TO COMPUTATION CHOSEN CHARACTERISTICS IN SIMPLE TRANSPORTATION SETS

The main aim of this article is the presentation of the way of making calculation with the application of the semi-Markov processes, with the continuous time t, where non-exponential distribution function is going to be applied. These type of calculations were published but the time there was approaching infinity. Engineering practice proves that the lack of solutions of this issue leads to obtainment of solutions that are encumbered with errors. This article is the first attempt of the error analysis.

SEMI-MARKOWSKA METODA WYZNACZANIA WYBRANYCH CHARAKTERYSTYK W PROSTYCH UKŁADACH TRANSPORTOWYCH

Celem artykułu jest przedstawienie sposobu wyznaczania podstawowych charakterystyk związanych z eksploatacją i niezawodnością systemów transportowych wykorzystując procesy semi-markowskie. Zazwyczaj w artykułach technicznych dokonuje się uproszczeń zakładając, że zjawiska „zawsze" mają charakter wykładniczy. Tymczasem rzeczywistość może być opisana poprzez inne, bardziej skomplikowane rozkłady gęstości prawdopodobieństwa występowania różnorodnych zjawisk. Wówczas typowe, oprogramowane metody obliczeniowe nie sprawdzają się, należy sięgnąć po zaawansowane narzędzia matematyczne, takie jak procesy semi-markowskie. Artykuł jest swego rodzaju dyskusją nad pojawiającymi się dylematami podczas wyznaczania charakterystyk niezawodności i eksploatacji przy nie wykładniczych funkcjach rozkładu prawdopodobieństwa przejść miedzy stanami.

1. INTRODUCTION

Dependability sub-functions like reliability and related measures, as availability, maintainability, failure rate, mean times, etc., are very important in design, development and life of real technical systems. While calculation dependability contributors for technical objects usually is assumed, that probabilities of transition between states or sojourn times' probabilities are exponential. Lack of information, too little number of samples or inaccurate assessment of data may cause that such assumption is abused. In some cases,

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when exponential distribution is assumed, there is also possibility to assess factors according to different distributions, like Weibull, Erlang, etc (Budny and Zajac, 2009)
Probabilities of transition between states and availability belong to the fundamental characteristic of reliability. While the discrete-time case can be obtained from the continuous one, by considering counting measure for discrete time points, we consider that it is important to give separately this case since an increasing interest is observed in practice for the discrete case, (Lisnianski and Levitin 2003, Zajac, 2007, Barbu and Limnios 2008, Chryssaphinou 2010, Valis and Vintr and Koucky, 2010). The discrete-time model, on one hand, is much simpler to handle numerically than the continuous-time one. On the other hand, it can used to handle numerically continuous-time formulated problems. So, for practical reliability problems it is better to work in discrete-time (Limnios, 2011).
The paper consist discussion on possibility and reasonability of carrying out calculation using Markov and semi-Markov methods on simple example with attempts to use continuous time in calculations. Discussion is based on prepared exponential and non-exponential sojourn times' probabilities. Valuation of methods is based on comparison availability and probabilities of transition values.

2. PATH'S ASSUMPTION

2.1. Markov approach

Let's make assumption that:

$P_1(t)$ - probability of sojourn time in state of serviceability in moment *t*;

$P_2(t)$ - probability of sojourn time in state of unserviceability in moment *t*;

$\lambda(t)$ - intensity of failures;

$\mu(t)$ - intensity of repairs.

Than matrix of intensities of transition

$$\Lambda = \begin{bmatrix} -\mu(t) & \mu(t) \\ \lambda(t) & -\lambda(t) \end{bmatrix} \tag{1}$$

Matrix can be described by graph like on figure 1.

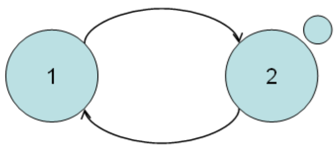

Figure. 1. Graph of state transition

Matrix is solved by set of Chapman-Kolmogorov equations:

$$P_1^{'}(t) = P_2(t)\,\lambda(t) - \mu(t)\,P_1(t)$$

$$P_2^{'}(t) = -P_2(t)\,\lambda(t) + \mu(t)\,P_1(t) \qquad (2)$$

$$P_1(t) + P_2(t) = 1, \qquad P_1(0) = 0;\;\; P_1(0) = 1.$$

When $\lambda(t) = \lambda$, $\mu(t) = \mu$, availability model of an object is represented by Markov process. Than can be obtained

$$P_2(t) = \frac{\lambda}{\lambda+\mu}\exp[-(\lambda+\mu)\,t\;] + \frac{\mu}{\lambda+\mu}$$

$$P_1(t) = 1 - P_2(t) = \frac{\lambda}{\lambda+\mu}\{1 - \exp[-(\lambda+\mu)\,t\;]\} \qquad (3)$$

In stationary set probabilities of states are given from formulas:

$$P_1 = \lim_{t\to\infty} P_1(t) = \frac{\lambda}{\mu+\lambda}$$

$$P_2 = \lim_{t\to\infty} P_2(t) = \frac{\mu}{\mu+\lambda} \qquad (4)$$

2.2. Semi Markov approach

There are three methods to define semi – Markov processes (Grabski, 2002, Grabski and Jazwinski 2003):
- by pair (***p***, **Q**(t)),
when: ***p*** – vector of initial distribution, **Q**(t) – matrix of distribution functions of transition times between states;
- by threes (***p***, ***P***, **F**(t)),
where: ***p*** – vector of initial distribution, ***P*** – matrix of transition probabilities, **F**(t) – matrix of distribution functions of sojourn times in state *i-th*, when *j-th* state is next;
- by threes (***p***, ***e***(t), **G**(t)),
where: ***p*** – vector of initial distribution, ***e***(t) – matrix of probabilities of transition between *i-th* and *j-th* states, when sojourn time in state *i-th* is x, **G**(t) – matrix of unconditional sojourn times distribution functions.
In particular example semi – Markov process is defined by (***p***, ***P***, **F**(t)).
Transient probabilities are one of the most important characteristics of semi – Markov processes. They are defined as conditional probabilities

$$P_{ij}(t) = P\{X(t) = j \mid X(0) = i\},\;\; i, j \in S \qquad (5)$$

Above probabilities obey Feller's equations (Grabski 2002, Grabski and Jazwinski 2003)

$$P_{ij}(t) = \delta_{ij}[1 - G_i(t)] + \sum_{k \in S} \int_0^t P_{kj}(t-x) dQ_{ik}(x), \tag{6}$$

Solution of that set of equations can be found by applying the Laplace – Stieltjes transformation. After that transformation the set takes form

$$\tilde{p}_{ij}(s) = \delta_{ij}[1 - \tilde{g}_i(s)] + \sum_{k \in S} \tilde{q}_{ik}(s) \tilde{p}_{kj}(s), \quad i, j \in S, \tag{7}$$

In matrix notation this set of equation has form

$$\tilde{\mathbf{p}}(s) = [I - \tilde{\mathbf{g}}(s)] + \tilde{\mathbf{q}}(s) \tilde{\mathbf{p}}(s), \tag{8}$$

hence

$$\tilde{\mathbf{p}}(s) = [I - \tilde{\mathbf{q}}(s)]^{-1} [I - \tilde{\mathbf{g}}(s)] . \tag{9}$$

3. CONDITIONS DETERMINATION FOR PARTICULAR EXAMPLE

Assumed system, presented on figure 1, consist of two states. Object can stay in reliability states from the set S (0,1), where:
0 – unserviceability state,
1 – serviceability state.

First state is described by random variable ζ_{p1}. The distribution function of random variable is

$$F_{\zeta p1}(t) = P\{\zeta_{p1} \le t\}, \ t \ge 0 .$$

Normal activities can be interrupted by failures. If there is known time, when the system is broken down, and that time is given by χ_p, then the distribution function of state "repair" is

$$F_{\chi_p}(t) = P\{\chi_p \le t\}, \ t \ge 0 .$$

The process can be described by semi – Markov process $\{X(t) : t \ge 0\}$ with the finite set of states $S_p = \{1, 2\}$. The kernel of the process is described by matrix

$$\boldsymbol{Q_p}(t) = \begin{bmatrix} 0 & Q_{01} \\ Q_{10} & 0 \end{bmatrix}, \tag{10}$$

Transition from 1-st state to 2-nd can be described by

$$Q_{p01}(t) = p_{01} F_{\zeta_{p1}}(t),$$

Transition from 2-th state to 1-st:

$$Q_{p01}(t) = P(\chi_p < t) = F_{\chi_p}(t).$$

The vector $\boldsymbol{p} = [p_1, p_2]$ is initial distribution of the process, in particular example $\boldsymbol{p} = [1,0]$.

4. DATA AND ASSUMPTIONS FOR CALCULATIONS

For needs of particular example two states set-up is prepared, which is mathematically described in chapter 2.1 by Markov process and chapter 3 by semi Markov process. Prepared data includes information on sojourn times during 100 points of time. Initial value of sojourn time of serviceability is equal to 10, after 100 observations decreases to value of 9.81. Each following number is lower on 0.01. This set can described simple technical object, that normal maintenance started before earlier and last observation didn't finish one. Sojourn time on unserviceability is constant and is equal to 1.

Collected data didn't allow for verify probabilities distribution. However the data allow to asses main parameters characterizing sample according to exponential and Weibull functions distributions.

In practice simplifications based on assuming exponential distribution often is like routine. Consequently values of availability or transient probabilities are calculated basing on Markov process, with usage of mean values of sojourn state times or states transitions. This method is described in charter 4.1.

Authors conducted calculation assuming, that size of the sample and its charter allow carrying out calculations basing on variable mean values of sojourn time. These variables are results of 10 mean values obtain from 10 separated intervals. Chapter 4.2 includes calculations based on variables mean values obtained from intervals. In charter 4.3 there is assumption, that transition from state 1 to state 2 is be described by Weibull function distribution, reverse transitions is exponential. Last calculations are done according to semi-Markov procedures, residual calculation represent markovian point of view. Factors necessary to carry out calculations are presented in Table 1.

Table 1. Distribution parameters for different distribution function

state 1	state 2
Parameter of exponential distribution (variant 1)	
λ=0.105	μ=1
Weibull and exponential distributions (variant 3)	
α=0.105	μ=1
β=1.1	

4.1 Markov calculation with one mean value

At first calculation has been done with assumption, that transient probabilities are exponential. Value of parameters is obtained from sample of 100. Mean time of sojourn time of state 1 is 9.52, intensity of transition between state 1 and 2 is 0.105, in reverse direction is constant and equal to 1. The distribution function of sojourn times and their Laplace – Stieltjes transformation take form:

$$F_{w1}(t) = 1 - e^{-0.105t} \ , \ f_{w1}*(t) = \frac{0.105}{s + 0.105},$$

$$F_{w2}(t) = 1 - e^{-t}, \ f_{w2}*(t) = \frac{1}{s+1}.$$

Then, kernel of the process is given by matrix

$$\boldsymbol{Q_p}(t) = \begin{bmatrix} 0 & 1 - e^{-0.105t} \\ 1 - e^{-t} & 0 \end{bmatrix}. \tag{8}$$

Matrices $\tilde{\mathbf{q}}(s)$ and $\tilde{\mathbf{g}}(s)$ have been determined according to equations (7) – (9). In considered example we obtain

$$\tilde{\mathbf{q}}(s) = \begin{bmatrix} 1 & \frac{0.105}{s + 0.105} \\ \frac{1}{s+1} & 1 \end{bmatrix}$$

However, taking into account recent experience (Zajac and Budny, 2009) calculations will be carried out with Markov model with time going to infinity. In this case intensities of transition given for particular example are presented in table 2.

Values P01 and P10 very quickly take stable values. In case of P10 it's after t=8. In particular example when sojourn time of state 1 decreases transient probability of transition to state 2 increases also.

Table 2. Intensities of transition in intervals

t	P00	P01	P10	P11
1	0.901	0.099	0.631	0.369
2	0.811	0.189	0.864	0.136
3	0.730	0.270	0.950	0.050
4	0.658	0.342	0.981	0.019
5	0.592	0.408	0.993	0.007
6	0.534	0.466	0.997	0.003
7	0.480	0.520	0.999	0.001
8	0.433	0.567	1	0
9	0.390	0.610	1	0
10	0.351	0.649	1	0
20	0.123	0.877	1	0
30	0.043	0.957	1	0

40	0.015	0.985	1	0
50	0.005	0.995	1	0

4.2. Variant with exponential distributions and constant intensities in intervals

Taking into account prepared data authors assumed that value of intensity of transition will be calculated for each 10 samples. Having 100 observations it gives 10 intervals with variable mean values sojourn time of state 1st. In this case calculation procedures are the same like in previous variant. Introducing variable mean values prescribed to intervals, uncertainty of evaluation is expected to be smaller. Values of intensities of transition in intervals are presented in table 3.

According to Markov calculation rules it is possible to obtain results presented in table 4. Parameter P0 can be treating as value of availability. There is also value of availability obtain in recent calculations. Parameter assigned by "*" represent availability results for previous example.

Table. 3. Results of states probabilities

t	λ	μ	P0	P1	P*
0	0.1000	1	0.9091	0.0909	0.9050
10	0.1005		0.9087	0.0913	
20	0.1015		0.9079	0.0921	
30	0.1025		0.9070	0.0930	
40	0.1035		0.9062	0.0938	
50	0.1046		0.9053	0.0947	
60	0.1056		0.9045	0.0955	
70	0.1067		0.9036	0.0964	
80	0.1077		0.9027	0.0973	
90	0.1088		0.9019	0.0981	
100	0.1099		0.9010	0.0990	

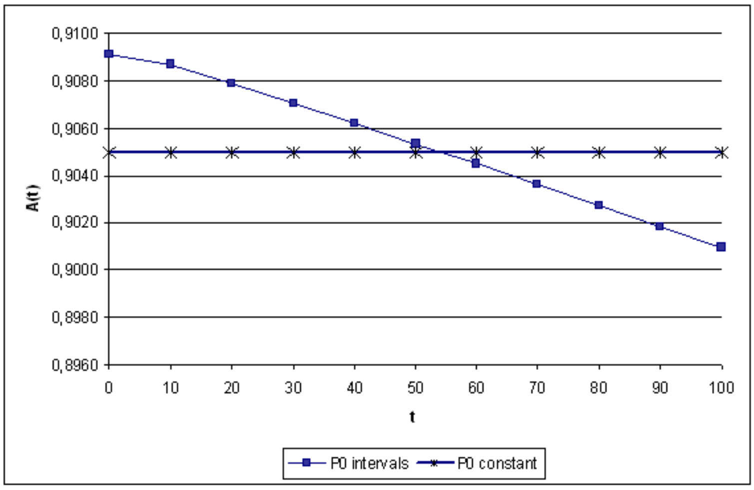

Figure 2. Comparison of values of availability obtained by Markov methods

On figure 2 representing results of comparison of two simple methods. It can be seen, that lowering number of intensities of transition causes degreasing value of transient probabilities and availability as time increasing. Calculating values of availability by first method can be treating as first estimation. Second way of calculating gives more detailed information.
Comparison values of transient probabilities P01 obtained using two methods of calculating didn't give serious differences. Sensitivity analysis didn't bring answer which method gives certain values.

4.3. Variant with Weibull distributions and semi-Markov calculations

In third variant of calculation it was decide, that sojourn time of state 1 is given by Weibull function distribution, for state 2 is exponential. According to table 2, collected data can be described by Weibull distributions. For particular calculations sojourn times distribution functions take form:

$$F_{b1}(x) = 1 - e^{-0.105t^{1.1}},$$

$$F_{b2}(x) = 1 - e^{-t}.$$

Derivative of Weibull distribution function (i.e. density function) is presented by

$$F'(t) = \lambda\alpha \cdot e^{-\lambda t^{\alpha}} \cdot t^{\alpha-1} \tag{16}$$

Laplace – Stieltjes transformations of Weibull distribution function can be obtained by using formula

$$f^*(t) = \int_0^\infty e^{-st} \cdot F'(t)\,dt = \int_0^\infty e^{-st}(1-e^{-\lambda t^\alpha})'dt = \\ = s\int_0^\infty e^{-st}(1-e^{-\lambda t^\alpha})dt - (1-e^{-\lambda \cdot 0^\alpha}) \tag{17}$$

Hence

$$f^*(t) = s\int_0^\infty e^{-st}(1-e^{-\lambda t^\alpha})dt = \\ = s\int_0^\infty e^{-st}dt - s\int_0^\infty e^{-st}\cdot e^{-\lambda t^\alpha}dt = 1 - s\int_0^\infty e^{-st}\cdot e^{-\lambda t^\alpha}dt \tag{18}$$

Using Maclaurin series for element "$e^{-\lambda t^\alpha}$", we obtain Laplace – Stieltjes transformation of the Weibull distribution function

$$f^*(t) = \lambda\frac{\alpha\cdot\Gamma(\alpha)}{s^\alpha} - \frac{\lambda^2}{2!}\frac{2\alpha\cdot\Gamma(2\alpha)}{s^{2\alpha}} + \frac{\lambda^3}{3!}\frac{3\alpha\cdot\Gamma(3\alpha)}{s^{3\alpha}} - ... = \sum_{n=1}^{\infty}\frac{\lambda^n}{n!}\frac{n\alpha\cdot\Gamma(n\alpha)}{S^{n\alpha}} \tag{19}$$

For considered example, Weibull distribution Laplace – Stieltjes transformations take form, respectively

$$f_{b1}^*(t) = \frac{1}{s+1}$$

$$f_{b2}^*(t) = \frac{0{,}1083}{s^{1{,}1}} - \frac{0{,}1254}{s^{2{,}2}} + \frac{0{,}0016}{s^{3{,}3}} - \frac{0{,}0002}{s^{4{,}4}} + ...$$

Matrices $\tilde{\mathbf{q}}(s)$ and $\tilde{\mathbf{g}}(s)$ have been determined according to equations (5) – (7). In considered example obtain

$$\boldsymbol{Q_p}(t) = \begin{bmatrix} 0 & 1-e^{-0.105t^{1.1}} \\ 1-e^{-t} & 0 \end{bmatrix},$$

$$\tilde{\mathbf{q}}(s) = \begin{bmatrix} 1 & \frac{0{,}1083}{s^{1{,}1}} - \frac{0{,}1254}{s^{2{,}2}} \\ \frac{1}{s+1} & 1 \end{bmatrix},$$

$$\tilde{\mathbf{g}}(s) = \begin{bmatrix} \frac{0,1083}{s^{1,1}} - \frac{0,1254}{s^{2,2}} & 0 \\ 0 & \frac{1}{s+1} \end{bmatrix}.$$

State probability can be obtained form formula:

$$P_0 = 0,9066203925 - 0,8132407848 \cdot e^{-1,181495882t}$$
$$-0,093379607500 \cdot e^{-0,222144t}$$

Figure 3 presents graphs of state probabilities.

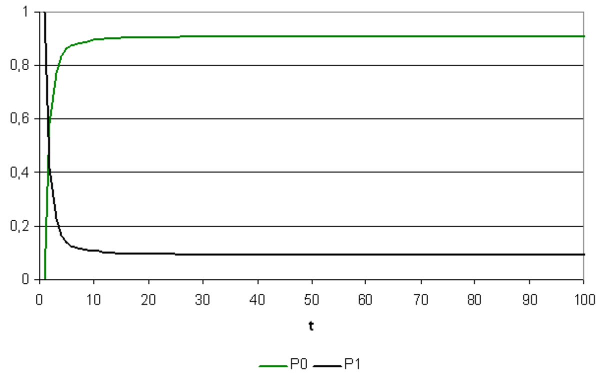

Figure 3. State probabilities graph with exponential and Weiubull functions.

Both functions distributions are non monotonic. Going to infinity they obtain constant values of P_1, P_2, calculated basing on Laplace transformation :

$$P_i = \lim_{s \to 0} s\tilde{P}_i(s), \quad i = 1,2,$$

or ergodic theory for semimarkov processes. In particular example:

$$P_1 = 0,9066, \qquad P_2 = 0,0934$$

5. CONCLUSIONS

Lack of information about type of distribution and routine assessment of exponential distribution can bring not accurate assumptions and consequently false results.

Semi - Markov processes allow for estimate crucial characteristics like availability or transient probabilities for objects, where distributions functions are discretional.

Usages of distribution functions other than exponential in case of semi Markov processes causes that further calculations are very complicated, but is possible to obtain certain results. Because of difficulties in calculation, profits from usage of semi – Markov processes are limited. However simple models can be computed.

Aim of future work is to continue research on applicability of semi Markov computation with different than exponential function distribution. Improvement of analytical results gives chance to prepare accurate software simulator in the future, simplify calculation and decrease level of uncertainties.

6. ACKNOLEGMENTS

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