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Stability of descriptor positive linear systems

Stabilność, dodatniość, układy singularne, pęk regularny.

Abstract

The purpose of the paper is to propose a new method for checking the positivity and asymptotic stability of descriptor continuous-time and discrete-time linear systems. The proposed method is based on the elementary operations applied to descriptor linear systems with regular pencils. The descriptor systems with regular pencils are transformed to equivalent standard systems which are used for checking of the asymptotic stability of descriptor systems.

STABILNOŚĆ SINGULARNYCH LINIOWYCH UKŁADÓW DODATNICH

Streszczenie

W pracy podano warunki konieczne i wystarczające dodatniości i stabilności asymptotycznej układów singularnych o pękach regularnych. Zaproponowano nową metodę sprawdzania dodatniości i asymptotycznej stabilności singularnych ciągłych i dyskretnych układów liniowych o pękach regularnych. Proponowana metoda jest oparta o działaniach elementarnych na pękach tych układów. W metodzie tej układy syngularne są redukowane do równoważnych postaci standardowych, które są wykorzystywane następnie do sprawdzania stabilności asymptotycznej tych układów singularnych.

1. INTRODUCTION

Descriptor (singular) linear systems have been considered in many papers and books (Dodig and Stosic, 2009; Dai, 1989; Fahmy and O'Reill, 1989; Gantmacher, 1960; Kaczorek, 2004; Kucera and Zagalak, 1988; Van Dooren, 1979; Virnik, 2008). The eigenvalues and invariants assignment by state and output feedbacks have been investigated in (Dodig and Stosic, 2009; Dai, 1989; Fahmy and O'Reill, 1989; Kaczorek, 2004) and the realization problem for singular positive continuous-time systems with delays in (Kaczorek, 2007b). The computation of Kronecker's canonical form of singular pencil has been analyzed in (Van Dooren, 1979). The positive linear systems with different fractional orders have been addressed in (Kaczorek, 2010; 2011b). Selected problems in theory of fractional linear systems has been given in monograph (Kaczorek, 2011b).

A dynamical system is called positive if its trajectory starting from any nonnegative initial state remains forever in the positive orthant for all nonnegative inputs. An overview of state of the art in positive theory is given in (Commalut and Marchand, 2006; Farina and Rinaldi, 2000; Kaczorek, 2002). Variety of models having positive behavior can be found in engineering, economics, social sciences, biology and medicine, etc..

Descriptor standard positive linear systems by the use of Drazin inverse has been addressed in (Bru et. all, 2003, 2000, 2002; Campbell et. all, 1976; Kaczorek, 1992). The shuffle algorithm has been applied to checking the positivity of descriptor linear systems in (Kaczorek, 2011a). The stability of positive descriptor systems has been investigated in (Virnik, 2008).

In this paper the new method based on the elementary operations for testing the positivity and asymptotic stability of the descriptor linear systems will be proposed.

The paper is organized as follows. In section 2 basic definitions and theorems concerning the standard positive continuous-time and discrete-time linear systems are recalled. The elementary operations are applied to checking the positivity and asymptotic stability of descriptor continuous-time linear systems in section 3 and of the descriptor discrete-time linear systems in section 4. Concluding remarks are given in section 5.

The following notation will be used: $\Re$ - the set of real numbers, $\Re^{n\times m}$ - the set of $n\times m$ real matrices, Z_+ - the set of nonnegative integers, $\Re_+^{n\times m}$ - the set of $n\times m$ matrices with nonnegative entries and $\Re_+^n = \Re_+^{n\times 1}$, M_n - the set of $n\times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), M_{ns} - the set of $n\times n$ asymptotically stable Metzler matrices, I_n - the $n\times n$ identity matrix.

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2. PRELIMINARIES

Consider the autonomous continuous-time linear system

$$\dot{x}(t) = Ax(t) \tag{2.1}$$

where $x(t) \in \Re^n$ is the state vector and $A \in \Re^{n \times n}$.

The system (2.1) is called (internally) positive if $x(t) \in \Re_+^n$, $t \geq 0$ for any initial conditions $x(0) = x_0 \in \Re_+^n$.

Theorem 2.1. (Farina, and Rinaldi, 2000; Kaczorek, 2002) The system (2.1) is positive if and only if

$$A \in M_n . \tag{2.2}$$

The positive system (2.1) is called asymptotically stable if

$$\lim_{t \to \infty} x(t) = 0 \text{ for all } x_0 \in \Re_+^n . \tag{2.3}$$

Theorem 2.2. (Farina, and Rinaldi, 2000; Kaczorek, 2002) The positive system (2.1) is asymptotically stable only if all diagonal entries of the matrix A are negative.

Let $A = [a_{ij}] \in \Re^{n \times n}$ be a Metzler matrix with negative diagonal entries ($a_{ii} < 0, \quad i = 1,...,n$).

Let define

$$A_n^{(0)} = A = \begin{bmatrix} a_{11}^{(0)} & \dots & a_{1,n}^{(0)} \\ \vdots & \dots & \vdots \\ a_{n,1}^{(0)} & \dots & a_{n,n}^{(0)} \end{bmatrix} = \begin{bmatrix} a_{11}^{(0)} & b_{n-1}^{(0)} \\ c_{n-1}^{(0)} & A_{n-1}^{(0)} \end{bmatrix}, \quad A_{n-1}^{(0)} = \begin{bmatrix} a_{22}^{(0)} & \dots & a_{2,n}^{(0)} \\ \vdots & \dots & \vdots \\ a_{n,2}^{(0)} & \dots & a_{n,n}^{(0)} \end{bmatrix},$$
$$b_{n-1}^{(0)} = [a_{12}^{(0)} \quad \dots \quad a_{1,n}^{(0)}], \quad c_{n-1}^{(0)} = \begin{bmatrix} a_{21}^{(0)} \\ \vdots \\ a_{n,1}^{(0)} \end{bmatrix} \tag{2.4a}$$

and

$$A_{n-k}^{(k)} = A_{n-k}^{(k-1)} - \frac{c_{n-k}^{(k-1)} b_{n-k}^{(k-1)}}{a_{k,k}^{(k-1)}} = \begin{bmatrix} a_{k+1,k+1}^{(k)} & \dots & a_{k+1,n}^{(k)} \\ \vdots & \dots & \vdots \\ a_{n,k+1}^{(k)} & \dots & a_{n,n}^{(k)} \end{bmatrix} = \begin{bmatrix} a_{k+1,k+1}^{(k)} & b_{n-k-1}^{(k)} \\ c_{n-k-1}^{(k)} & A_{n-k-1}^{(k)} \end{bmatrix},$$
$$A_{n-k-1}^{(k)} = \begin{bmatrix} a_{k+2,k+2}^{(k)} & \dots & a_{k+2,n}^{(k)} \\ \vdots & \dots & \vdots \\ a_{n,k+2}^{(k)} & \dots & a_{n,n}^{(k)} \end{bmatrix}, \quad b_{n-k-1}^{(k)} = [a_{k+1,k+2}^{(k)} \quad \dots \quad a_{k+1,n}^{(k)}], \quad c_{n-k-1}^{(k)} = \begin{bmatrix} a_{k+2,k+1}^{(k)} \\ \vdots \\ a_{n,k+1}^{(k)} \end{bmatrix} \tag{2.4b}$$

for k = 1,…,n – 1. It is well-known that using elementary operations we may reduce the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1,n} \\ a_{21} & a_{22} & \dots & a_{2,n} \\ \vdots & \vdots & \dots & \vdots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} \end{bmatrix} \tag{2.5}$$

to the lower triangular form

$$\tilde{A} = \begin{bmatrix} \tilde{a}_{11} & 0 & \dots & 0 \\ \tilde{a}_{21} & \tilde{a}_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{n,1} & \tilde{a}_{n,2} & \dots & \tilde{a}_{n,n} \end{bmatrix} . \tag{2.6}$$

Theorem 2.3. (Farina, and Rinaldi, 2000; Kaczorek, 2002, 2011b) The positive system (2.1) (the matrix $A \in \Re^{n \times n}$) is a asymptotically stable if and only if one of the following equivalent conditions is satisfied:

1) All coefficients of the characteristic polynomial

$$\det[I_n s - A] = s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0 \tag{2.7}$$

are positive, i.e. $a_i > 0$ for i = 1,…,n – 1.

2) All principal minors $\Delta_i, \quad i = 1,...,n$ of the matrix $-A = [-a_{ij}]$ are positive, i.e.

$$\Delta_1 = -a_{11} > 0, \quad \Delta_2 = \begin{vmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{vmatrix} > 0, \dots, \Delta_n = \det[-A] > 0 \tag{2.8}$$

3) The diagonal entries of the matrices (2.4)

$$A_{n-k}^{(k)} \text{ for } k = 1,\dots,n-1 \tag{2.9}$$

are negative.

4) The diagonal entries of the lower triangular matrix (2.6) are negative, i.e.

$$\tilde{a}_{kk} < 0 \text{ for } k = 1,\ldots,n. \tag{2.10}$$

Consider the autonomous discrete-time linear system

$$x_{i+1} = \bar{A}x_i,\ i \in Z_+ = \{0,1,...\} \tag{2.11}$$

where $x_i \in \Re^n$ is the state vector and $\bar{A} \in \Re^{n\times n}$.

The system (2.11) is called (internally) positive if $x_i \in \Re_+^n$, $i \in Z_+$ for any initial conditions $x_0 \in \Re_+^n$.

Theorem 2.4. The system (2.11) is positive if and only if

$$\bar{A} \in \Re_+^{n\times n}. \tag{2.12}$$

The positive system (2.11) is called asymptotically stable if

$$\lim_{i\to\infty} x_i = 0 \text{ for all } x_0 \in \Re_+^n \tag{2.13}$$

Theorem 2.5. The positive system (2.11) is asymptotically stable if and only if one of the conditions of Theorem 2.3 is satisfied for matrix $A = \bar{A} - I_n$.

3. DESCRIPTOR CONTINUOUS-TIME LINEAR SYSTEMS

Consider the descriptor autonomous continuous-time linear system

$$E\dot{x}(t) = Ax(t),\ x(0) = x_0 \tag{3.1}$$

where $x(t) \in \Re^n$ is the state vectors and $E, A \in \Re^{n\times n}$. It is assumed that $\det E = 0$ and the pair (E, A) or the pencil $Es - A$ is regular, i.e.

$$\text{p(s)} = \det[Es - A] \neq 0 \text{ for some } s \in C \text{ (the field of complex number).} \tag{3.2}$$

Definition 3.1. The systems (3.1) is called (internally) positive if for every consistent nonnegative initial condition $x_0 \in R_+^n$, $x(t) \in R_+^n$ for $t \geq 0$.

The following elementary row operations will be used [Kaczorek 2002]:

1) Multiplication of the i-th row (column) by a real nonzero number c. This operation will be denoted by $L[i\times c]$ ($R[i\times c]$).
2) Addition to the i-th row (column) of the j-th row (column) multiplied by a real number c. This operation will be denoted by $L[i + j\times c]$ ($R[i + j\times c]$).
3) Interchange of the i-th and j-th rows (columns). This operation will be denoted by $L[i, j]$ ($R[i, j]$).

A matrix $Q \in \Re_+^{n\times n}$ is called monomial if its every row and column contains only one positive entry and its remaining entries are zero. The inverse matrix Q^{-1} of the monomial matrix Q has only nonnegative entries, i.e. $Q^{-1} \in \Re_+^{n\times n}$ [Kaczorek 2007a].

It is assumed that using elementary row and column operations it is possible to reduce the pair (E, A) to the form

$$P[Es - A]Q = \bar{E}s - \bar{A} \tag{3.3a}$$

where

$$\bar{E} = \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix},\ \bar{A} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix},\ \bar{A}_{11} \in M_{n_1},\ \bar{A}_{22} \in M_{n_2},\ \bar{A}_{12} \in \Re_+^{n_1\times n_2},\ \bar{A}_{21} \in \Re_+^{n_2\times n_1},$$
$$n_1 = \text{rank } E,\quad n_2 = n - n_1 \tag{3.3b}$$

and $P \in \Re^{n\times n}$ is a matrix of elementary row operations and $Q \in \Re_+^{n\times n}$ is a monomial matrix of elementary column operations.

The matrix P can be obtained by performing the elementary row operations and the matrix Q by performing the elementary column operations on identity matrix I_n, respectively [Kaczorek 2007a].

Note that if Q is a monomial matrix then $\bar{x}(t) = Q^{-1}x(t) \in \Re_+^n$, $t \geq 0$ for every $x(t) \in \Re_+^n$, $t \geq 0$ since $Q^{-1} \in \Re_+^{n\times n}$.

Theorem 3.1. The descriptor continuous-time linear system (3.1) is positive and asymptotically stable if and only if there exists a pair of elementary operations matrices (P, Q) satisfying (3.3) such that the coefficients of the polynomials

$$p_1(s) = \det\begin{bmatrix} I_{n_1}s - \bar{A}_{11} & -\bar{A}_{12} \\ -\bar{A}_{21} & -\bar{A}_{22} \end{bmatrix} = \bar{a}_{n_1}s^{n_1} + \bar{a}_{n_1-1}s^{n_1-1} + ... + \bar{a}_1 s + \bar{a}_0 \tag{3.4a}$$

and

$$p_2(s) = \det[I_{n_2}s - \bar{A}_{22}] = s^{n_2} + a_{n_2-1}s^{n_2-1} + ... + a_1 s + a_0 \tag{3.4b}$$

are positive, i.e. $\bar{a}_j > 0,\ j = 0,1,...,n_1$ and $a_i > 0,\ i = 0,1,...,n_2 - 1$.

Proof. It is well-known [Kaczorek 2002, 2007a] that the coefficients $a_i > 0,\ i = 0,1,...,n_2 - 1$ of (3.4b) only if the matrix A_{22} has only nonnegative diagonal entries and it can be reduced by elementary row operations to the matrix $-I_{n_2}$, i.e.

$$P_2\bar{A}_{22} = -I_{n_2} \tag{3.5}$$

where $P_2 \in \Re_+^{n_2\times n_2}$ is a matrix of elementary row operations. Using (3.3) and (3.5) we obtain

$$\begin{bmatrix} I_{n_1} & 0 \\ 0 & P_2 \end{bmatrix}\begin{bmatrix} I_{n_1}s-\bar{A}_{11} & -\bar{A}_{12} \\ -\bar{A}_{21} & -\bar{A}_{22} \end{bmatrix} = \begin{bmatrix} I_{n_1}s-\bar{A}_{11} & -\bar{A}_{12} \\ -P_2\bar{A}_{21} & I_{n_2} \end{bmatrix}. \tag{3.6}$$

Premultiplying the matrix (3.6) by the matrix

$$\begin{bmatrix} I_{n_1} & \bar{A}_{12} \\ 0 & I_{n_2} \end{bmatrix} \in \Re_+^{n\times n} \tag{3.7}$$

we get

$$\begin{bmatrix} I_{n_1} & \bar{A}_{12} \\ 0 & I_{n_2} \end{bmatrix}\begin{bmatrix} I_{n_1}s-\bar{A}_{11} & -\bar{A}_{12} \\ -P_2\bar{A}_{21} & I_{n_2} \end{bmatrix} = \begin{bmatrix} I_{n_1}s-\bar{A}'_{11} & 0 \\ -P_2\bar{A}_{21} & I_{n_2} \end{bmatrix} \tag{3.8}$$

where $\bar{A}'_{11} = \bar{A}_{11} + \bar{A}_{12}P_2\bar{A}_{21} \in M_{n_1}$ since $\bar{A}_{11} \in M_{n_1}$ and $\bar{A}_{12}P_2\bar{A}_{21} \in \Re_+^{n_1\times n_1}$.

Note that

$$\det[I_{n_1}s-\bar{A}'_{11}] = \det\begin{bmatrix} I_{n_1}s-\bar{A}'_{11} & 0 \\ -P_2\bar{A}_{21} & I_{n_2} \end{bmatrix} = \det\begin{bmatrix} I_{n_1} & \bar{A}_{12} \\ 0 & I_{n_2} \end{bmatrix}\det\begin{bmatrix} I_{n_1}s-\bar{A}_{11} & -\bar{A}_{12} \\ -P_2\bar{A}_{21} & I_{n_2} \end{bmatrix}$$

$$= \det\begin{bmatrix} I_{n_1} & 0 \\ 0 & P_2 \end{bmatrix}\det\begin{bmatrix} I_{n_1}s-\bar{A}_{11} & -\bar{A}_{12} \\ -\bar{A}_{21} & -\bar{A}_{22} \end{bmatrix} = p\det\begin{bmatrix} I_{n_1}s-\bar{A}_{11} & -\bar{A}_{12} \\ -\bar{A}_{21} & -\bar{A}_{22} \end{bmatrix}$$

where $p = \det P_2$ is nonzero constant.

Therefore, the coefficients $\bar{a}_j > 0,\ \ j = 0,1,...,n_1$ of (3.4a) if and only if the matrix $\bar{A}'_{11}$ is asymptotically stable Metzler matrix, i.e. $\bar{A}'_{11} \in M_{n_1s}$ and

$$\dot{\bar{x}}_1(t) = \bar{A}'_{11}\,\bar{x}_1(t) \tag{3.9a}$$

and

$$\bar{x}_2(t) = P_2\bar{A}_{21}\bar{x}_1(t) \tag{3.9b}$$

where

$$\bar{x}(t) = \begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \end{bmatrix} = Q^{-1}x\,.$$

From (3.9) we have $\bar{x}_1(t) \in \Re_+^{n_1}$, $\bar{x}_2(t) \in \Re_+^{n_2}$, $t \ge 0$ since $\bar{A}'_{11} \in M_{n_1}$, $P_2\bar{A}_{21} \in \Re_+^{n_2\times n_1}$ and

$$\lim_{t\to\infty}\bar{x}_1(t) = e^{\bar{A}'_{11}t}\bar{x}_1(0) = 0 \ \text{ for } \ \bar{x}_1(0) \in \Re_+^{n_1}$$

and

$$\lim_{t\to\infty}\bar{x}_2(t) = \lim_{t\to\infty}P_2\bar{A}_{21}\bar{x}_1(t) = 0\,.$$

This completes the proof. □

From (3.2), (3.3a) and (3.4a) we obtain

$$p(s) = \det[Es-A] = \det\{P^{-1}[\bar{E}s-\bar{A}]Q^{-1}\} = \det[P^{-1}Q^{-1}]\det[\bar{E}s-\bar{A}] = kp_1(s) \tag{3.10}$$

where $k = \det[P^{-1}Q^{-1}] = \det[QP]^{-1} = \dfrac{1}{\det PQ}$ and $p_1(s)$ is defined by (3.4a).

From Theorem 3.1 and (3.10) we have the following corollary.

Corollary 3.1. The descriptor system (3.1) is positive and asymptotically stable only if all coefficients of the polynomial (3.2) are nonzero and have the same sign, i.e. all coefficients are positive if $k > 0$ and negative if $k < 0$.

Note that to check the asymptotic stability of descriptor system also the remaining conditions (2) – 4)) of Theorem 2.3 can be used.

Example 3.1. Check the positivity and asymptotic stability of the descriptor system (3.1) with the matrices

$$E = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & -2 & 0 & -4 \\ 0 & 1 & 0 & 2 \\ 0 & -1 & 0 & -2 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 1 & -2 \\ 0 & 4 & -7 & 7 \\ 1 & -1 & 2 & -3 \\ -5 & 2 & -1 & 3 \end{bmatrix}. \tag{3.11}$$

Performing on the array

$$\begin{matrix} E & A \end{matrix} = \begin{matrix} 0 & 0 & 0 & 1 & & 0 & 1 & 1 & -2 \\ 0 & -2 & 0 & -4 & & 0 & 4 & -7 & 7 \\ 0 & 1 & 0 & 2 & & 1 & -1 & 2 & -3 \\ 0 & -1 & 0 & -2 & & -5 & 2 & -1 & 3 \end{matrix} \qquad (3.12)$$

the following elementary operations $L[2+3\times 2]$, $L[4+3\times 1]$, $L[3+1\times(-2)]$, $L[2,3]$ and $R[1,4]$ we obtain

$$\begin{matrix} \bar{E} & \bar{A} \end{matrix} = \begin{matrix} 1 & 0 & 0 & 0 & & -2 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & & 1 & -3 & 0 & 1 \\ 0 & 0 & 0 & 0 & & 1 & 2 & -3 & 2 \\ 0 & 0 & 0 & 0 & & 0 & 1 & 1 & -4 \end{matrix} \qquad (3.13a)$$

or equivalently

$$\bar{E} = PEQ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \bar{A} = PAQ = \begin{bmatrix} -2 & 1 & 1 & 0 \\ 1 & -3 & 0 & 1 \\ 1 & 2 & -3 & 2 \\ 0 & 1 & 1 & -4 \end{bmatrix} \qquad (3.13b)$$

where

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \qquad (3.13c)$$

Using (3.13b) and (3.4) we obtain

$$p_1(s) = \det\begin{bmatrix} I_{n_1}s - \bar{A}_{11} & -\bar{A}_{12} \\ -\bar{A}_{21} & -\bar{A}_{22} \end{bmatrix} = \begin{vmatrix} s+2 & -1 & -1 & 0 \\ -1 & s+3 & 0 & -1 \\ -1 & -2 & 3 & -2 \\ 0 & -1 & -1 & 4 \end{vmatrix} = 10s^2 + 41s + 18 \qquad (3.14a)$$

and

$$p_2(s) = \det[I_{n_2}s - \bar{A}_{22}] = \begin{vmatrix} s+3 & -2 \\ -1 & s+4 \end{vmatrix} = s^2 + 7s + 10\,. \qquad (3.14b)$$

By Theorem 3.1 the descriptor system with (3.11) is positive and asymptotically stable since the coefficients of the polynomials (3.14) are positive.

In this case det PQ = 1 and

$$p(s) = \det[Es - \bar{A}_{22}] = \begin{vmatrix} 0 & -1 & -1 & s+2 \\ 0 & -2s-4 & 7 & -4s-7 \\ -1 & s+1 & -2 & 2s+3 \\ 5 & -s-2 & 1 & -2s-3 \end{vmatrix} = 10s^2 + 41s + 18 = p_1(s)\,.$$

Example 3.2. Consider the descriptor system (3.1) with the matrices

$$E = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \qquad (3.15)$$

Performing on the array

$$\begin{matrix} E & A \end{matrix} = \begin{matrix} 0 & 1 & 0 & & 1 & 0 & 0 \\ 0 & -1 & 0 & & 0 & 1 & 0 \\ 0 & 0 & 1 & & 0 & 1 & 0 \end{matrix} \qquad (3.16)$$

the following elementary operations $L[2+1\times 1]$, $L[2,3]$, $R[1,2]$ and $R[2,3]$ we obtain

$$\begin{matrix} \bar{E} & \bar{A} \end{matrix} = \begin{matrix} 1 & 0 & 0 & & 0 & 0 & 1 \\ 0 & 1 & 0 & & 1 & 0 & 0 \\ 0 & 0 & 0 & & 1 & 0 & 1 \end{matrix}, \quad n_1 = 2 \qquad (3.17)$$

Using (3.17) and (3.4) we obtain

$$p_1(s) = \det\begin{bmatrix} I_{n_1}s - \bar{A}_{11} & -\bar{A}_{12} \\ -\bar{A}_{21} & -\bar{A}_{22} \end{bmatrix} = \begin{vmatrix} s & 0 & -1 \\ -1 & s & 0 \\ -1 & 0 & -1 \end{vmatrix} = -s^2 - s \qquad (3.18a)$$

and

$$p_2(s)=\det[I_{n_2}s-\overline{A}_{22}]=s-1\,. \tag{3.18b}$$

Therefore by Theorem 3.1 the descriptor system with (3.15) is unstable.

4. DESCRIPTOR DISCRETE-TIME SYSTEMS

Consider the descriptor discrete-time linear system

$$\hat{E}x_{i+1}=\hat{A}x_i\,,\ i\in Z_+=\{0,1,...\} \tag{4.1}$$

where $x_i\in\Re^n$ is the state vector and $\hat{E},\hat{A}\in\Re^{n\times n}$. It is assumed that $\det\hat{E}=0$ and the pair $(\hat{E},\hat{A})$ or the pencil $\hat{E}z-\hat{A}$ is regular, i.e.

$$\det[\hat{E}z-\hat{A}]\neq 0 \ \text{ for some } z\in C\,. \tag{4.2}$$

Definition 4.1. The system (4.1) is called (internally) positive if for every consistent nonnegative initial condition $x_0\in R_+^n$, $i\in Z_+$, $x_i\in R_+^n$, $i\in Z_+$.

It is assumed that using elementary row and column operations it is possible to reduce the pair $(\hat{E},\hat{A})$ to the form

$$\hat{P}[\hat{E}z-\hat{A}]\hat{Q}=\tilde{E}z-\tilde{A} \tag{4.3a}$$

where

$$\tilde{E}=\begin{bmatrix} I_{n_1} & 0\\ 0 & 0\end{bmatrix},\ \ \tilde{A}=\begin{bmatrix}\tilde{A}_{11} & \tilde{A}_{12}\\ \tilde{A}_{21} & \tilde{A}_{22}\end{bmatrix},\ \ \tilde{A}_{11}\in\Re_+^{n_1\times n_1},\ \ \tilde{A}_{22}\in\Re_+^{n_2\times n_2},\ \ \tilde{A}_{12}\in\Re_+^{n_1\times n_2},\ \ \tilde{A}_{21}\in\Re_+^{n_2\times n_1},$$
$$n_1=\operatorname{rank}\hat{E},\ \ n_2=n-n_1 \tag{4.3b}$$

and $\hat{P}\in\Re^{n\times n}$ is a matrix of elementary row operations and $\hat{Q}\in\Re_+^{n\times n}$ is a monomial matrix of elementary column operations.

Theorem 4.1. The descriptor discrete-time linear system (4.1) is positive and asymptotically stable if and only if there exists a pair of elementary operations matrices $(\hat{P},\hat{Q})$ satisfying (4.3) such that the conditions of the polynomials

$$\hat{p}_1(z)=\det\begin{bmatrix} I_{n_1}(z+1)-\tilde{A}_{11} & -\tilde{A}_{12}\\ -\tilde{A}_{21} & -\tilde{A}_{22}\end{bmatrix}=\hat{a}_{n_1}z^{n_1}+\hat{a}_{n_1-1}z^{n_1-1}+...+\hat{a}_1z+\hat{a}_0 \tag{4.4a}$$

and

$$\hat{p}_2(z)=\det[I_{n_2}(z+1)-\tilde{A}_{22}]=z^{n_2}+\tilde{a}_{n_2-1}z^{n_2-1}+...+\tilde{a}_1z+\tilde{a}_0 \tag{4.4b}$$

are positive, i.e. $\hat{a}_j>0,\ \ j=0,1,...,n_1$ and $\tilde{a}_i>0,\ \ i=0,1,...,n_2-1$.

Proof. Premultiplying the pencil $[\hat{E}z-\hat{A}]$ by the matrix $\hat{P}$, defining the new state vector

$$\tilde{x}_i=\begin{bmatrix}\tilde{x}_{1i}\\ \tilde{x}_{2i}\end{bmatrix}=\hat{Q}^{-1}x_i\,,\ \tilde{x}_{1i}\in\Re_+^{n_1}\,,\ \tilde{x}_{2i}\in\Re_+^{n_2} \tag{4.5}$$

and using (4.3) we obtain

$$\tilde{x}_{1i+1}=\tilde{A}_{11}\tilde{x}_{1i}+\tilde{A}_{12}\tilde{x}_{2i} \tag{4.6a}$$

$$0=\tilde{A}_{21}\tilde{x}_{1i}+\tilde{A}_{22}\tilde{x}_{2i}\,. \tag{4.6b}$$

The coefficients of (4.4b) are positive if and only if the matrix $\tilde{A}_{22}-I_{n_2}$ is an asymptotically stable Metzler matrix. In this case from (4.6a) we have

$$\tilde{x}_{2i}=-\tilde{A}_{22}^{-1}\tilde{A}_{21}\tilde{x}_{1i}\in\Re_+^{n_2}\ \text{ if }\ \tilde{x}_{1i}\in\Re_+^{n_1}\,,\ i\in Z_+ \tag{4.7}$$

since $-\tilde{A}_{22}^{-1}\in\Re_+^{n_2\times n_2}$ [Kaczorek 2002].

Substitution of (4.7) into (4.6a) yields

$$\tilde{x}_{1i+1}=\tilde{A}'_{11}\,\tilde{x}_{1i} \tag{4.8a}$$

where

$$\tilde{A}'_{11}=\tilde{A}_{11}-\tilde{A}_{12}\tilde{A}_{22}^{-1}\tilde{A}_{21}\in\Re_+^{n_1\times n_2}\,. \tag{4.8b}$$

From the equality

$$\begin{bmatrix} I_{n_1} & -\tilde{A}_{12}\tilde{A}_{22}^{-1}\\ 0 & I_{n_2}\end{bmatrix}\begin{bmatrix}\tilde{A}_{11} & \tilde{A}_{12}\\ \tilde{A}_{21} & \tilde{A}_{22}\end{bmatrix}=\begin{bmatrix}\tilde{A}_{11}-\tilde{A}_{12}\tilde{A}_{22}^{-1}\tilde{A}_{21} & 0\\ \tilde{A}_{21} & \tilde{A}_{22}\end{bmatrix}. \tag{4.9}$$

it follows that

$$\det\begin{bmatrix} I_{n_1}(z+1)-\tilde{A}_{11} & -\tilde{A}_{12}\\ -\tilde{A}_{21} & -\tilde{A}_{22}\end{bmatrix}=\det[I_{n_1}(z+1)-\tilde{A}'_{11}]\det[-\tilde{A}_{22}]\,. \tag{4.10}$$

By Theorem 2.5 the positive descriptor system (4.1) is asymptotically stable if and only if the coefficients of the polynomial (4.4a) are positive, since $\det[-\tilde{A}_{22}] > 0$. □

Example 4.1. Consider the descriptor system (4.1) with the matrices

$$\hat{E} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ -2 & -2 & 0 \end{bmatrix}, \quad \hat{A} = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -2 \end{bmatrix}. \tag{4.11}$$

Performing on the array

$$\hat{E} \quad \hat{A} = \begin{matrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ -2 & -2 & 0 \end{matrix} \quad \begin{matrix} 0.5 & 0.5 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -2 \end{matrix} \tag{4.12}$$

the following elementary operations $L[3+2\times 1]$, $L[3+1\times 2]$ and $L[2\times 0.5]$ we obtain

$$\tilde{E} \quad \tilde{A} = \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{matrix} \quad \begin{matrix} 0.5 & 0.5 & 0 \\ 0 & 0 & 0.5 \\ 0 & 0 & -1 \end{matrix}, \; n_1 = 2. \tag{4.13}$$

Using (4.4) and (4.13) we obtain

$$\hat{p}_1(z) = \det\begin{bmatrix} I_{n_1}(z+1) - \tilde{A}_{11} & -\tilde{A}_{12} \\ -\tilde{A}_{21} & -\tilde{A}_{22} \end{bmatrix} = \begin{vmatrix} z+0.5 & -0.5 & 0 \\ 0 & z+1 & -0.5 \\ 0 & 0 & 1 \end{vmatrix} = z^2 + 1.5z + 0.5 \tag{4.14a}$$

and

$$\hat{p}_2(z) = \det[I_{n_2}(z+1) - \tilde{A}_{22}] = z + 2. \tag{4.14b}$$

Therefore, by Theorem 4.1 the descriptor system with (4.11) is positive and asymptotically stable.

5. CONCLUDING REMARKS

A method based on the elementary operations for checking of the positivity and stability of descriptor continuous-time and discrete-time linear systems have been proposed. Using the elementary operations the descriptor systems with regular pencils have been transformed to equivalent standard linear systems. Necessary and sufficient conditions for the positivity and asymptotic stability of the descriptor systems have been established (Theorems 3.1 and 4.1). The effectiveness of the proposed method has been illustrated on numerical examples. The considerations can be also extended for fractional descriptor linear systems (Kaczorek, 2011b).

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