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DECOMPOSITION OF THE PAIRS (*A*,*B*) AND (*A*,*C*) OF POSITIVE DISCRETE-TIME LINEAR SYSTEMS

A new test for checking the reachability (observability) of positive discrete-time linear systems is proposed. Conditions are established under which the unreachable pair (A,B) and the unobservable pair (A,C) of positive discrete-time system can be decomposed into reachable and unreachable parts and observable and unobservable parts, respectively. It is shown that the transfer matrix of the positive system is equal to the transfer matrix of its reachable (observable) part.

DEKOMPOZYCJI PAR (A,B) I (A,C) DODATNICH UKŁADÓW DYSKRETNYCH

W pracy zaproponowano nowe kryteria badania osiągalności (obserwowalności) dodatnich liniowych układów dyskretnych. Podano warunki przy spełnieniu których para nieosiągalna (A,B) oraz para nieobserwowalna (A,C) dodatniego układu dyskretnego może być zdekomponowana na część osiągalna i nieosiągalną oraz odpowiednio na część obserwowalna i nieobserwowalną. Wykazano, że macierz transmitancji układu dodatniego jest równa macierzy transmitancji tylko części osiągalnej oraz odpowiednio tylko części obserwowalnej.

1. INTRODUCTION

In positive systems inputs, state variables and outputs take only non-negative values. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive linear behavior can be found in engineering, management science, economics, social sciences, biology and medicine, etc. An overview of state of the art in positive linear theory is given in the monographs [2, 3].

The notions of controllability and observability and the decomposition of linear systems have been introduced by Kalman [7, 8]. Those notions are the basic concepts of the modern control theory [1, 6, 9, 10, 5]. They have been also extended to positive linear systems [2, 3].

In this paper the idea of Kalman's decomposition theorem will be extended to positive discrete-time linear systems. Conditions will be established for the decomposition of the

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pair (A,B) into reachable and unreachable parts and of the pair (A,C) into observable and unobservable parts.

The paper is organized as follows. In section 2 the basic definitions and theorem concerning reachability and observability of positive discrete-time linear systems are recalled. New test for checking the reachability of positive discrete-time linear systems is proposed in section 3. The main result of the paper is given in section 4 and 5. In section 4 the conditions for decomposition of the pair (A,B) into reachable and unreachable parts are proposed and in section 5 the conditions for decomposition of the pair (A,C) into observable and unobservable parts. Concluding remarks are given in section 6.

2. PRELIMINARIES

The set of $n\times m$ real matrices will be denoted by $\Re^{n\times m}$ and $\Re^{n}:=\Re^{n\times 1}$. The set of $m\times n$ real matrices with nonnegative entries will be denoted by $\Re_{+}^{m\times n}$ and $\Re_{+}^{n}:=\Re_{+}^{n\times 1}$. The set of nonnegative integers will be denoted by Z_{+} and the $n\times n$ identity matrix by I_n.

Consider the linear discrete-time systems

$$\begin{aligned} x_{i+1} &= Ax_i + Bu_i, \quad i\in Z_+ \\ y_i &= Cx_i + Du_i \end{aligned} \tag{2.1}$$

where $x_i\in\Re^{n}$, $u_i\in\Re^{m}$, $y_i\in\Re^{p}$ are the state, input and output vectors and $A\in\Re^{n\times n}$, $B\in\Re^{n\times m}$, $C\in\Re^{p\times n}$, $D\in\Re^{p\times m}$.

Definition 2.1. The system (2.1) is called (internally) positive if and only if $x_i\in\Re_{+}^{n}$, and $y_i\in\Re_{+}^{p}$, $i\in Z_+$ for every $x_0\in\Re_{+}^{n}$, , and any input sequence $u_i\in\Re_{+}^{m}$, $i\in Z_+$.

Theorem 2.1. [2, 3] The system (2.1) is (internally) positive if and only if

$$A\in\Re_{+}^{n\times n},\quad B\in\Re_{+}^{n\times m},\quad C\in\Re_{+}^{p\times n},\quad D\in\Re_{+}^{p\times m}. \tag{2.2}$$

Definition 2.2. The positive system (2.1) is called reachable in q steps if there exists an input sequence $u_i\in\Re_{+}^{m}$, $i=0,1,\ldots,q-1$ which steers the state of the system from zero ($x_0=0$) to any given final state $x_f\in\Re_{+}^{n}$, i.e. $x_q=x_f$.

Let e_i, $i=1,\ldots,n$ be the ith column of the identity matrix I_n. A column ae_i for $a>0$ is called the monomial column.

Theorem 2.2. [2, 3] The positive system (2.1) is reachable in q steps if and only if the reachability matrix

$$R_q=[B \quad AB \quad \ldots \quad A^{q-1}B]\in\Re_{+}^{n\times qm} \tag{2.3}$$

contains n linearly independent monomial columns.

Theorem 2.3. [2, 3] The positive system (2.1) is reachable in q steps only if the matrix

$$[B \quad A] \tag{2.4}$$

contains n linearly independent monomial columns.

Definition 2.3. The positive systems (2.1) is called observable in q steps if it is possible to find unique initial state $x_0 \in \Re_+^n$ of the system knowing its input sequence $u_i \in \Re_+^m$, $i = 0,1,...,q-1$ and its corresponding output sequence $y_i \in \Re_+^p$, $i = 0,1,...,q-1$.

Theorem 2.4. [2, 3] The positive systems (2.1) is observable in q steps if and only if the observability matrix

$$O_q = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{q-1} \end{bmatrix} \in \Re_+^{qp\times n} \tag{2.5}$$

contains n linearly independent monomial rows.

Theorem 2.5. [2, 3] The positive system (2.1) is observable in q steps only if the matrix

$$\begin{bmatrix} C \\ A \end{bmatrix} \tag{2.6}$$

contains n linearly independent monomial rows.

3. NEW TEST FOR CHECKING THE REACHABILITY OF POSITIVE LINEAR SYSTEMS

In this section a new test for checking the reachability of the pair

$$A = [A_1 \quad A_2 \quad ... \quad A_n] \in \Re_+^{n\times n},\ B = [B_1 \quad B_2 \quad ... \quad B_m] \in \Re_+^{n\times m},$$
$$A_i = \begin{bmatrix} a_{1i} \\ \vdots \\ a_{ni} \end{bmatrix},\ i = 1,...,n;\ B_j = \begin{bmatrix} b_{1j} \\ \vdots \\ b_{nj} \end{bmatrix},\ j = 1,...,m; \tag{3.1}$$

will be proposed.

First we assume that $m = 1$ and $B = B_1$. Let B_1 be a monomial column with positive entry b_{i_1}. In the new test for checking the reachability of the pair (3.1) a crucial role will play the following procedure of finding a sequence of linearly independent monomial columns (compare with [4]).

Procedure.

The monomial column B_1 with $b_{i_1} > 0$, $i_1 \in (1,...,n)$ is the first element of the sequence. If the column $A_{i_1} (= AB_{i_1})$ is monomial and linearly independent $(a_{i_2 i_1} > 0, i_1 \neq i_2)$ then it belongs to the sequence B_1, A_{i_1} . If the column is not monomial or $i_1 = i_2$ it does not belong to the sequence and the procedure stop. In this case the pair (3.1) is not reachable. If the column $A_{i_2} (= AA_{i_1} = A^2 B_{i_1})$ is monomial and linearly independent of the columns B_1 and A_{i_1} $(a_{i_3 i_2} > 0, i_3 \neq i_2, i_3 \neq i_1)$ then it belongs to the sequence B_1, A_{i_1}, A_{1_2} .

Continuing the procedure we may find the sequence of linearly independent monomial columns

$$B_1, A_{i_1}, ..., A_{1_k} . \tag{3.2}$$

The positive linear system (2.1) or equivalently the pair (3.1) is reachable in $q = n$ steps if and only the sequence (3.2) contains n elements ($k = n - 1$).

Therefore, the following theorem has been proposed.

Theorem 3.1. The single-input positive system (2.1) is reachable in n steps if and only if using Procedure it is possible to find n linearly independent monomial columns (3.2) for $k = n - 1$.

Example 3.1. Check the reachability of the pair

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \tag{3.3}$$

For the pair (3.3) the sequence (3.2) has the form

$$B_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad A_{i_1} = A_1 = AB_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad A_{i_2} = A_2 = A^2 B_1 = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}. \tag{3.4}$$

The sequence (3.4) contains three linearly independent monomial columns and the pair by Theorem 3.1 is reachable in $q = n = 3$ steps.

Example 3.2. Check the reachability of the pair

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \tag{3.5}$$

Note that the pair satisfies the necessary condition of reachability (Theorem 2.3).

In this case the sequence (3.2) for the pair (3.5) contains only one monomial column $B_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ since the column $A_1 = AB_1 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ is not monomial. Therefore, the pair is not reachable.

If $m > 1$ then using Procedure sequentially for each linearly independent monomial column of the matrix B we may find the sequence of the linearly independent monomial columns

$$B_1, A_{i_{11}}^{(1)}, ..., A_{i_{1k_1}}^{(1)}, B_2, A_{i_{21}}^{(2)}, ..., A_{i_{2k_2}}^{(2)}, ..., B_m, A_{i_{m1}}^{(m)}, ..., A_{i_{mk_m}}^{(m)}. \tag{3.6}$$

The positive linear systems (2.1) for $m > 1$ is reachable in q steps ($q \le n$) if and only if the sequence (3.6) contains n linearly independent columns for $k_i \le q$, $i = 1, ..., m$.

Therefore, the following theorem has been proved.

Theorem 3.2. The multi-input positive system (2.1) is reachable in q steps ($q \le n$) if and only if using Procedure it is possible to find n linearly independent monomial columns (3.6) for $k_i \le q$, $i = 1, ..., m$.

Example 3.3. Check the reachability of the pair

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}. \tag{3.7}$$

For the first monomial column of the matrix B we have

$$B_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}. \tag{3.8}$$

The first two columns of (3.8) are only linearly independent monomial columns. For the second monomial column of the matrix B we obtain also only two linearly independent monomial columns

$$B_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}. \tag{3.9}$$

In this case the sequence (3.6) has the form

$$B_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}. \tag{3.10}$$

By Theorem 3.2 the positive system with (3.7) is reachable in $q = 2$ steps.

4. DECOMPOSITION OF POSITIVE PAIR (*A*,*B*)

Let the pair (*A*,*B*) of the positive systems (2.1) be unreachable but the sequence (3.6) contains at least one monomial column. First we shall consider the single-input ($m = 1$) system and we shall assume that the matrix $B \in \Re_+^{n\times m}$ is monomial, otherwise the positive system is unreachable for any matrix $A \in \Re_+^{n\times n}$.

Let us assume that the reachability matrix

$$R_n = [B \quad AB \quad \dots \quad A^{n-1}B] \in \Re_+^{n\times n} \tag{4.1}$$

of the positive system (2.1) has $n_1 < n$ linearly independent monomial columns

$$P_1 = B, \quad P_2 = AB, \quad P_{n_1} = A^{n_1-1}B\,. \tag{4.2}$$

It is always possible to chose $n_2 = n - n_1$ linearly independent monomial columns

$$P_{n_1+1}, P_{n_1+2}, \dots, P_n \tag{4.3}$$

which are orthogonal to the columns (4.2).

The matrix

$$P = [P_1 \quad \dots \quad P_{n_1} \quad P_{n_1+1} \quad \dots \quad P_n] \tag{4.4}$$

is monomial and its inverse P^{-1} is equal to P^T, where *T* denotes the transpose.

It is assumed that the following condition

$$P_k^T A P_{n_1} = 0 \text{ for } k = n_1 + 1, \dots, n \tag{4.5}$$

is satisfied.

Note that (4.5) holds if AP_{n_1} is a linear combination of the monomial columns $P_1, \dots, P_{n_1}$.

We shall show that if the condition (4.5) is satisfied then using the matrix (4.4) we can reduce the pair (*A*, *B*) to the following form

$$\overline{A} = P^{-1}AP = \begin{bmatrix} \overline{A}_1 & \overline{A}_{12} \\ 0 & \overline{A}_2 \end{bmatrix}, \quad \overline{B} = P^{-1}B = \begin{bmatrix} \overline{B}_1 \\ 0 \end{bmatrix} \tag{4.6}$$

where the pair

$$\overline{A}_1 \in \Re_+^{n_1\times n_1}, \quad \overline{B}_1 \in \Re_+^{n_1} \tag{4.7}$$

is reachable and the pair $\overline{A}_2 \in \Re_+^{n_2\times n_2}$, $\overline{B}_2 = 0 \in \Re_+^{n_2}$ is unreachable.

From (4.2) we have

$$AP = [AP_1 \quad AP_2 \quad \dots \quad AP_{n_1} \quad AP_{n_1+1} \quad \dots \quad AP_n] = [P_2 \quad P_3 \quad \dots \quad P_{n_1} \quad AP_{n_1} \quad \dots \quad AP_n] \tag{4.8}$$

Taking into account that (4.2) are orthogonal to (4.3) and using (4.8) we obtain

$$P^{-1}AP = P^T AP = \begin{bmatrix} P_1^T \\ \vdots \\ P_{n_1}{}^T \\ P_{n_1+1}{}^T \\ \vdots \\ P_n{}^T \end{bmatrix} [P_2 \quad P_3 \quad \dots \quad P_{n_1} \quad AP_{n_1} \quad \dots \quad AP_n] = \begin{bmatrix} \overline{A}_1 & \overline{A}_{12} \\ 0 & \overline{A}_2 \end{bmatrix} \tag{4.9a}$$

where

$$\overline{A}_1 = \begin{bmatrix} P_1^T P_2 & \dots & P_1^T P_{n_1} & P_1^T AP_{n_1} \\ \vdots & \dots & \vdots & \vdots \\ P_{n_1}{}^T P_2 & \dots & P_{n_1}{}^T P_{n_1} & P_{n_1}{}^T AP_{n_1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \dots & 0 & \overline{a}_1 \\ 1 & 0 & \dots & 0 & \overline{a}_2 \\ 0 & 1 & \dots & 0 & \overline{a}_3 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & \overline{a}_{n_1} \end{bmatrix} \in \Re_+^{n_1 \times n_1},$$

$$\overline{a}_k = P_k{}^T AP_{n_1}, \quad k = 1, \dots, n_1;$$

$$\overline{A}_{12} = \begin{bmatrix} P_1^T AP_{n_1+1} & \dots & P_1^T AP_{n-1} & P_1^T AP_n \\ \vdots & \dots & \vdots & \vdots \\ P_{n_1}{}^T AP_{n_1+1} & \dots & P_{n_1}{}^T AP_{n-1} & P_{n_1}{}^T AP_n \end{bmatrix} \in \Re_+^{n_1 \times n_2}, \tag{4.9b}$$

$$\overline{A}_2 = \begin{bmatrix} P_{n_1+1}{}^T AP_{n_1+1} & \dots & P_{n_1+1}{}^T AP_{n-1} & P_{n_1+1}{}^T AP_n \\ \vdots & \dots & \vdots & \vdots \\ P_n{}^T AP_{n_1+1} & \dots & P_n{}^T AP_{n-1} & P_n{}^T AP_n \end{bmatrix} \in \Re_+^{n_2 \times n_2}$$

and

$$\begin{bmatrix} P_{n_1+1}{}^T P_2 & \dots & P_{n_1+1}{}^T P_{n_1} & P_{n_1+1}{}^T AP_{n_1} \\ \vdots & \dots & \vdots & \vdots \\ P_n{}^T P_2 & \dots & P_n{}^T P_{n_1} & P_n{}^T AP_{n_1} \end{bmatrix} = \begin{bmatrix} 0 & \dots & 0 \\ \vdots & \dots & \vdots \\ 0 & \dots & 0 \end{bmatrix} \in \Re^{n_2 \times n_1} \tag{4.9c}$$

since (4.5) holds.

Taking into account that

$$P_k{}^T B = \begin{cases} 1 & \text{for} \quad k = 1 \\ 0 & \text{for} \quad k = 2, \dots, n \end{cases} \tag{4.10}$$

we obtain

$$\bar{B} = P^{-1}B = P^T B = \begin{bmatrix} P_1^T \\ \vdots \\ P_n^T \end{bmatrix} B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} \bar{B}_1 \\ 0 \end{bmatrix}. \tag{4.11}$$

Using (4.9) and (4.11) it is easy to verify that the pair $(\bar{A}_1, \bar{B}_1)$ is reachable since

$$[\bar{B}_1 \quad \bar{A}_1\bar{B}_1 \quad \ldots \quad \bar{A}_1^{n_1-1}\bar{B}_1] = I_{n_1} \tag{4.12}$$

Therefore the following theorem has been proved.

Theorem. 4.1. Let the positive system (2.1) be unreachable but the matrix (4.1) has n_1 linearly independent monomial columns and the assumption (4.5) be satisfied. Then the pair (A,B) of the system can be reduced to the form (4.6) by the use of the similarity transformation with monomial matrix (4.4). Moreover, the positive pair $(\bar{A}_1, \bar{B}_1)$ is reachable.

Example 4.1. Consider the positive system (2.1) with the matrices

$$A = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \tag{4.13}$$

The pair is unreachable since the reachability matrix

$$R_4 = [B \quad AB \quad A^2B \quad A^3B] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{4.14}$$

has only two linearly independent monomial columns $P_1 = B$ and $P_2 = AB$.
In this case the monomial matrix (4.4) has the form

$$P = [P_1 \quad P_2 \quad P_3 \quad P_4] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.15}$$

and the assumption (4.5) is satisfied since

$$\begin{bmatrix} P_3^T \\ P_4^T \end{bmatrix} AP_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 2 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{4.16}$$

Using (4.6) we obtain

$$\overline{A} = P^{-1}AP = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 2 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \overline{A}_1 & \overline{A}_{12} \\ 0 & \overline{A}_2 \end{bmatrix} \tag{4.17}$$

and

$$\overline{B} = P^{-1}B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \overline{B}_1 \\ 0 \end{bmatrix}. \tag{4.18}$$

The pair

$$\overline{A}_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \overline{B}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{4.19}$$

is reachable since $[\overline{B}_1 \quad \overline{A}_1\overline{B}_1] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Example 4.2. Consider the positive system (2.1) with the matrices

$$A = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \tag{4.20}$$

The pair is unreachable since the reachability matrix

$$R_4 = [B \quad AB \quad A^2B \quad A^3B] = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{4.21}$$

has only two linearly independent monomial columns $P_1 = B$ and $P_2 = AB$.

In this case the monomial matrix (4.4) has also the form (4.15) but the assumption (4.5) is not satisfied since

$$\begin{bmatrix} P_3^T \\ P_4^T \end{bmatrix} AP_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 2 \\ 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{4.22}$$

Using (4.6) we obtain

$$\overline{A} = P^{-1}AP = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 2 \\ 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{4.23}$$

and

$$\overline{B} = P^{-1}B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{4.24}$$

In this case the matrices (4.23) and (4.24) has not the form (4.6).

Now let us consider the multi-input ($m > 1$) positive system (2.1). We shall assume that the matrix $B \in \Re_+^{n \times m}$ has at least one monomial column; otherwise the positive system is not reachable for any matrix $A \in \Re_+^{n \times n}$.

Let the reachability matrix

$$R_n = [B \quad AB \quad \dots \quad A^{n-1}B] \in \Re_+^{n \times mn} \tag{4.25}$$

has $n_1 < n$ linearly independent monomial columns

$$P_1, \quad P_2, \dots, \quad P_{n_1} \tag{4.26}$$

The columns of (4.26) are chosen as follows. Let the columns $B_{i_1}, B_{i_2}, \dots, B_{i_k}$ ($k \le m$) of the matrix B are linearly independent monomial columns. We choose from the sequence

$$AB_{i_1}, \dots, AB_{i_k}, A^2B_{i_1}, \dots, A^2B_{i_k}, \dots, A^{n-1}B_{i_1}, \dots, A^{n-1}B_{i_k} \tag{4.27}$$

such monomial columns which are linearly independent from the previously chosen monomial columns.

It is always possible to chose $n_2 = n - n_1$ linearly independent monomial columns

$$P_{n_1+1}, P_{n_1+2}, \dots, P_n \tag{4.28}$$

which are orthogonal to the columns (4.26).

Let the monomial matrix P have the form

$$P = [P_{i_1} \quad \dots \quad P_{i_1 d_1} \quad P_{i_2} \quad \dots \quad P_{i_2 d_2} \quad \dots \quad P_{i_k d_k} \quad P_{n_1+1} \quad \dots \quad P_n] = [P_1 \quad P_2 \quad \dots \quad P_n] \tag{4.29a}$$

where

$$P_{i_1} = B_{i_1}, ..., P_{i_1 d_1} = A^{d_1 - 1} B_{i_1}, P_{i_2} = B_{i_2}, ..., P_{i_2 d_2} = A^{d_2 - 1} B_{i_2}, ..., P_{i_k d_k} = A^{d_k - 1} B_{i_k} \quad (4.29b)$$

and d_i $(i = 1, ..., k)$ are some natural numbers.

Taking into account that

$$P^{-1} = P^T = \begin{bmatrix} P_1^T \\ \vdots \\ P_n^T \end{bmatrix} \text{ and } P_i^T P_j = \begin{cases} 1 & \text{for} \quad i = j \\ 0 & \text{for} \quad i \neq j \end{cases} \quad (4.30)$$

we obtain

$$\overline{A} = P^{-1} A P = \begin{bmatrix} P_1^T \\ \vdots \\ P_n^T \end{bmatrix} [A P_1 \quad AP_2 \quad ... \quad AP_n] = \begin{bmatrix} \overline{A}_1 & \overline{A}_{12} \\ 0 & \overline{A}_2 \end{bmatrix} \quad (4.31a)$$

where

$$\overline{A}_1 = \begin{bmatrix} P_1^T AP_1 & ... & P_1^T AP_{n_1} \\ \vdots & ... & \vdots \\ P_{n_1}^T AP_1 & ... & P_{n_1}^T AP_{n_1} \end{bmatrix}, \quad \overline{A}_{12} = \begin{bmatrix} P_1^T AP_{n_1+1} & ... & P_1^T AP_n \\ \vdots & ... & \vdots \\ P_{n_1}^T AP_{n_1+1} & ... & P_{n_1}^T AP_n \end{bmatrix},$$

$$\overline{A}_2 = \begin{bmatrix} P_{n_1+1}^T AP_{n_1+1} & ... & P_{n_1+1}^T AP_n \\ \vdots & ... & \vdots \\ P_n^T AP_{n_1+1} & ... & P_n^T AP_n \end{bmatrix} \quad (4.31b)$$

and

$$\overline{B} = P^{-1} B = \begin{bmatrix} P_1^T \\ \vdots \\ P_n^T \end{bmatrix} B = \begin{bmatrix} \overline{B}_1 \\ 0 \end{bmatrix}, \quad \overline{B}_1 = \text{blockdiag}[\overline{B}_1 \quad ... \quad \overline{B}_{i_k}], \quad B_i = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad i = i_1, ..., i_k \quad (4.31c)$$

if

$$P_k^T AP_j = 0 \quad \text{for} \quad k = n_1 + 1, ..., n; \quad j = 1, ..., n_1 \quad (4.32)$$

where the pair $(\overline{A}_1, \overline{B}_1)$ is reachable and the pair $(\overline{A}_2, \overline{B}_2 = 0)$ is unreachable.

Therefore, the following theorem has been proved.

Theorem 4.2. Let the positive system (2.1) be unreachable but the matrix (4.25) has n_1 linearly independent monomial columns and the assumption (4.32) be satisfied. Then the pair (A, B) of the system can be reduced to the form (4.31) by the use of the similarity transformation with the monomial matrix (4.29). Moreover the positive pair $(\overline{A}_1, \overline{B}_1)$ is reachable.

Example 4.3. Consider the positive system (2.1) with the matrices

$$A=\begin{bmatrix}1&0&0\\0&1&1\\0&0&2\end{bmatrix},\quad B=\begin{bmatrix}0&1\\1&0\\0&1\end{bmatrix}. \tag{4.33}$$

The pair is unreachable since the reachability matrix

$$R_3=[B \quad AB \quad A^2B]=\begin{bmatrix}0&1&0&1&0&1\\1&0&1&1&1&3\\0&1&0&2&0&4\end{bmatrix} \tag{4.34}$$

has only one monomial column $P_1=\begin{bmatrix}0\\1\\0\end{bmatrix}$.

In this case the monomial matrix (4.29) has the form

$$P=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix} \tag{4.35a}$$

and the assumption (4.32) is satisfied since

$$\begin{bmatrix}P_2^T\\P_3^T\end{bmatrix}AP_1=\begin{bmatrix}1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&1\\0&0&2\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}. \tag{4.35b}$$

Using (4.31), (4.33) and (4.35a) we obtain

$$\overline{A}=P^{-1}AP=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&1\\0&0&2\end{bmatrix}\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}=\begin{bmatrix}1&0&1\\0&1&0\\0&0&2\end{bmatrix}=\begin{bmatrix}\overline{A}_1&\overline{A}_{12}\\0&\overline{A}_2\end{bmatrix} \tag{4.36a}$$

and

$$\overline{B}=P^{-1}B=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}0&1\\1&0\\0&1\end{bmatrix}=\begin{bmatrix}1&0\\0&1\\0&1\end{bmatrix}=\begin{bmatrix}\overline{B}_1&0\\0&\overline{B}_2\end{bmatrix}. \tag{4.36b}$$

The positive pair $\overline{A}_1 = [1]$, $\overline{B}_1 = [1]$ is reachable and the positive pair $\overline{A}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $\overline{B}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is unreachable since $[\overline{B}_2 \quad \overline{A}_2\overline{B}_2] = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$.

Let

$$\overline{x}_i = P^{-1}x_i = \begin{bmatrix} \overline{x}_i^{(1)} \\ \overline{x}_i^{(2)} \end{bmatrix}, \quad \overline{x}_i^{(1)} \in \Re^{n_1}, \quad \overline{x}_i^{(2)} \in \Re^{n_2} \tag{4.37}$$

be a new state vector and

$$y_i = Cx_i + Du_i = CPP^{-1}x_i + Du_i = \overline{C}_1\overline{x}_i^{(1)} + \overline{C}_2\overline{x}_i^{(2)} + Du_i \tag{4.38}$$

where

$$CP = [\overline{C}_1 \quad \overline{C}_2], \quad \overline{C}_1 \in \Re^{p\times n_1}, \quad \overline{C}_2 \in \Re^{p\times n_2}. \tag{4.39}$$

Definition 4.1. The positive subsystem

$$\overline{x}_i^{(1)} = \overline{A}_1\overline{x}_i^{(1)} + \overline{B}_1 u_i \tag{4.40a}$$

$$y_i^{(1)} = \overline{C}_1\overline{x}_i^{(1)} + Du_i \tag{4.40b}$$

is called the reachable part of the system (2.1).

Theorem 4.3. The transfer matrix of the positive system (2.1)

$$T(z) = C[I_n z - A]^{-1}B + D \tag{4.41}$$

is equal to the transfer matrix of its reachable part (4.40)

$$T_1(z) = \overline{C}_1[I_{n_1} z - \overline{A}_1]^{-1}\overline{B}_1 + D \tag{4.42}$$

i.e. $T(z) = T_1(z)$.

Proof. Using (4.41), (4.6) and (4.39) we obtain

$$\begin{aligned} T(z) &= C[I_n z - A]^{-1}B + D = CPP^{-1}[I_n z - A]^{-1}PP^{-1}B + D = \\ &= \overline{C}[I_n z - PAP^{-1}]^{-1}\overline{B} + D = [\overline{C}_1 \quad \overline{C}_2]\begin{bmatrix} I_{n_1}z - \overline{A}_1 & -\overline{A}_{12} \\ 0 & I_{n_2}z - \overline{A}_2 \end{bmatrix}^{-1}\begin{bmatrix} \overline{B}_1 \\ 0 \end{bmatrix} + D \\ &= [\overline{C}_1 \quad \overline{C}_2]\begin{bmatrix} [I_{n_1}z - \overline{A}_1]^{-1} & * \\ 0 & [I_{n_2}z - \overline{A}_2]^{-1} \end{bmatrix}\begin{bmatrix} \overline{B}_1 \\ 0 \end{bmatrix} + D = \overline{C}_1[I_{n_1}z - \overline{A}_1]^{-1}\overline{B}_1 + D = T_1(z) \end{aligned} \tag{4.43}$$

Therefore, the transfer matrix (4.42) represents only reachable part of the positive system. □

5. DECOMPOSITION OF THE POSITIVE PAIR (*A*,*C*)

Definition 5.1. The positive system

$$\begin{aligned} x_{i+1} &= A^T x_i + C^T u_i, \quad i \in Z_+ \\ y_i &= B^T x_i + Du_i \end{aligned} \tag{5.1}$$

(the matrices *A*, *B*, *C*, *D* are the same as of (2.1)) is called the dual positive system with respect to the system (2.1).

Theorem 5.1. The positive system (2.1) is reachable in *q* steps if and only if the positive dual system (5.1) is observable in *q* steps.

Proof. By Theorem 2.2 the positive system (2.1) is reachable in *q* steps if and only if the reachability matrix

$$R_q = [B \quad AB \quad \ldots \quad A^{q-1}B] \tag{5.2}$$

contains *n* linearly independent monomial columns.
Note that the transpose matrix (5.2)

$$R_q^{\ T} = \begin{bmatrix} B^T \\ B^T A^T \\ \vdots \\ B^T (A^T)^{q-1} \end{bmatrix} \tag{5.3}$$

contains *n* linearly independent monomial rows if and only if the matrix (5.2) contains *n* linearly independent monomial columns.
The matrix (5.3) is the observability matrix of the positive system (5.1). By Theorem 2.4 the positive system (5.1) is observable in *q* steps if and only if the positive system (2.1) is reachable in *q* steps. □

Therefore, for testing the observability of the positive system (2.1) we can use the reachability conditions for dual positive system. The duality can be also used in the decomposition of the positive pair (*A*,*C*). Let the pair (*A*,*C*) of the positive single-output ($p = 1$) system (2.1) be unobservable but the matrix *C* be a monomial (otherwise the system is unobservable for any matrix $A \in \Re_+^{n \times n}$).

Let us assume that the observability matrix

$$O_n = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \in \Re_+^{n \times n} \tag{5.4}$$

has $n_1 < n$ linearly independent monomial rows

$$Q_1 = C, \quad Q_2 = CA, \quad ..., \quad Q_{n_1} = CA^{n_1-1}. \tag{5.5}$$

It is always possible to choose $n_2 = n - n_1$ linearly independent monomial rows

$$Q_{n_1+1}, \quad Q_{n_1+2}, ..., Q_n \tag{5.6}$$

which are orthogonal to the rows (5.5).
The matrix

$$Q = \begin{bmatrix} Q_1 \\ \vdots \\ Q_{n_1} \\ Q_{n_1+1} \\ \vdots \\ Q_n \end{bmatrix} \tag{5.7}$$

is monomial and its inverse $Q^{-1} = Q^T$.
We shall show that if

$$Q_{n_1} A Q_k{}^T = 0 \text{ for } k = n_1 + 1, ..., n \tag{5.8}$$

holds then using (5.7) we can reduce the pair (*A*,*C*) to the from

$$\hat{A} = QAQ^{-1} = \begin{bmatrix} \hat{A}_1 & 0 \\ \hat{A}_{21} & \hat{A}_2 \end{bmatrix}, \; \hat{C} = CQ^{-1} = [\hat{C}_1 \quad 0] \tag{5.9}$$

where the pair $(\hat{A}_1, \hat{C}_1)$ is observable and the pair $(\hat{A}_2, \hat{C}_2 = 0)$ is unobservable.
From (5.5) we have

$$QA = \begin{bmatrix} Q_1 A \\ Q_2 A \\ \vdots \\ Q_{n_1-1} A \\ Q_{n_1} A \\ \vdots \\ Q_n A \end{bmatrix} = \begin{bmatrix} Q_2 \\ Q_3 \\ \vdots \\ Q_{n_1} \\ Q_{n_1} A \\ \vdots \\ Q_n A \end{bmatrix}. \tag{5.10}$$

Taking into account that (5.6) are orthogonal to (5.5) and using (5.8) we obtain

$$\hat{A} = QAQ^T = \begin{bmatrix} Q_2 \\ Q_3 \\ \vdots \\ Q_{n_1} \\ Q_{n_1}A \\ \vdots \\ Q_n A \end{bmatrix} [Q_1^T \quad Q_2^T \quad \dots \quad Q_{n_1}^T \quad \dots \quad Q_n^T] = \begin{bmatrix} \hat{A}_1 & 0 \\ \hat{A}_{21} & \hat{A}_2 \end{bmatrix} \tag{5.11}$$

where

$$\hat{A}_1 = \begin{bmatrix} Q_2 Q_1^T & \dots & Q_2 Q_{n_1}^T \\ \vdots & \dots & \vdots \\ Q_{n_1} A Q_1^T & \dots & Q_{n_1} A Q_{n_1}^T \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ \hat{a}_1 & \hat{a}_2 & \hat{a}_3 & \dots & \hat{a}_{n_1} \end{bmatrix} \in \Re_+^{n_1 \times n_1},$$

$$\hat{a}_k = Q_{n_1} A Q_k^T, \quad k = 1, \dots, n_1;$$

$$\hat{A}_{21} = \begin{bmatrix} Q_{n_1+1} A Q_1^T & \dots & Q_{n_1+1} A Q_{n_1}^T \\ \vdots & \dots & \vdots \\ Q_n A Q_1^T & \dots & Q_n A Q_{n_1}^T \end{bmatrix} \in \Re_+^{n_2 \times n_1}, \quad \hat{A}_2 = \begin{bmatrix} Q_{n_1+1} A Q_{n_1+1}^T & \dots & Q_{n_1+1} A Q_n^T \\ \vdots & \dots & \vdots \\ Q_n A Q_{n_1+1}^T & \dots & Q_n A Q_n^T \end{bmatrix} \in \Re_+^{n_2 \times n_2} \tag{5.12}$$

Taking into account that

$$CQ_k^T = \begin{cases} 1 & \text{for} \quad k = 1 \\ 0 & \text{for} \quad k = 2, \dots, n \end{cases} \tag{5.13}$$

we obtain

$$\hat{C} = CQ^T = [\hat{C}_1 \quad 0], \quad \hat{C}_1 = [1 \quad 0 \quad \dots \quad 0] \in \Re_+^{1 \times n_1}. \tag{5.14}$$

Using (5.12) and (5.14) it is easy to verify that the pair $(\hat{A}_1, \hat{B}_1)$ is observable since

$$\begin{bmatrix} \hat{C}_1 \\ \hat{C}_1 \hat{A}_1 \\ \vdots \\ \hat{C}_1 \hat{A}_1^{n-1} \end{bmatrix} = I_{n_1} \tag{5.15}$$

Therefore, the following theorem has been proved.

Theorem. 5.2. Let the positive system (2.1) be unobservable but the matrix (5.1) has n_1 linearly independent monomial rows and the assumption (5.8) be satisfied. Then the pair (A,C) of the system can be reduced to the form (5.9) by the use of the similarity transformation with monomial matrix (5.7). Moreover, the positive pair $(\hat{A}_1, \hat{C}_1)$ is observable.

Example 5.1. Consider the positive system (2.1) with the matrices

$$A = \begin{bmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 1 \end{bmatrix}, \quad C = [0 \quad 1 \quad 0 \quad 0] \tag{5.16}$$

The pair is unobservable since the observability matrix

$$O_4 = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{5.17}$$

has only two linearly independent monomial rows $(n_1 = 2)$ $Q_1 = C$ and $Q_2 = CA$. In this case the monomial matrix (5.7) has the form

$$Q = \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{5.18}$$

and the assumption (5.8) is satisfied since

$$Q_2 A [Q_3^T \quad Q_4^T] = [0 \quad 0 \quad 1 \quad 0] \begin{bmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} = [0 \quad 0]. \tag{5.19}$$

Using (5.9) and (5.18) we obtain

$$\hat{A} = QAQ^T = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} \hat{A}_1 & 0 \\ \hat{A}_{21} & \hat{A}_2 \end{bmatrix} \tag{5.20a}$$

and

$$\hat{C} = CQ^T = [0 \quad 1 \quad 0 \quad 0]\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = [1 \quad 0 \quad 0 \quad 0] = [\hat{C}_1 \quad 0] \tag{5.20b}$$

The pair

$$\hat{A}_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \hat{C}_1 = [1 \quad 0] \tag{5.21a}$$

is observable since

$$\begin{bmatrix} \hat{C}_1 \\ \hat{C}_1\hat{A}_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{5.21b}$$

Note that the pair (5.16) is the dual pair to (4.13) and the results can be obtained by the duality principle.

In a similar way the considerations can be extended for the multi-input ($p > 1$) positive systems (2.1).

Let the observability matrix (5.4) has $n_1 < n$ linearly independent monomial rows and the rows $C_{j_1}, C_{j_2}, ..., C_{j_l}$ $(l \le p)$ of the matrix $C \in \Re_+^{p\times n}$ are linearly independent monomial rows. Then from the sequence

$$C_{j_1}A, ..., C_{j_l}A, C_{j_1}A^2, ..., C_{j_l}A^2, ..., C_{j_1}A^{n-1}, ..., C_{j_l}A^{n-1} \tag{5.22}$$

we may choose monomial rows which are linearly independent from the previously chosen monomial rows.

Let

$$Q_1, Q_2, ..., Q_{n_1} \tag{5.23}$$

be the linearly independent monomial rows of the matrix (5.4). Then it is possible to choose $n_2 = n - n_1$ linearly independent monomial rows

$$Q_{n_1+1}, Q_{n_1+2}, ..., Q_n \tag{5.24}$$

which are orthogonal to rows (5.23).

Let the monomial matrix Q^T have the form

$$Q^T = [Q_{j_1}^T \quad ... \quad Q_{j_1\bar{d}_1}^T \quad Q_{j_2}^T \quad ... \quad Q_{j_2\bar{d}_2}^T \quad ... \quad Q_{j_l\bar{d}_l}^T \quad Q_{n_1+1}^T \quad ... \quad Q_n^T] \tag{5.25a}$$

where

$$Q_{j_1} = C_{j_1}, ..., Q_{j_1\bar{d}_1} = C_{j_1}A^{\bar{d}_1-1}, Q_{j_2} = C_{j_2}, ..., Q_{j_2\bar{d}_2} = C_{j_2}A^{\bar{d}_2-1}, ..., Q_{j_l\bar{d}_l} = C_{j_l}A^{\bar{d}_l-1} \tag{5.25b}$$

and $\bar{d}_j$ $(j = 1, ..., l)$ are some natural numbers.

Taking into account that

$$Q_i Q_j^T = \begin{cases} 1 & \text{for} \quad i = j \\ 0 & \text{for} \quad i \neq j \end{cases} \tag{5.26}$$

and the assumption

$$Q_k A Q_j^T = 0 \text{ for } k = n_1 + 1, ..., n; \ j = n_1 + 1, ..., n_1 . \tag{5.27}$$

We can prove in a similar way as in the case $p = 1$ the following theorem.

Theorem 5.3. Let the positive system (2.1) be unobservable but the matrix (5.4) has n_1 linearly independent monomial rows and the assumption (5.27) be satisfied. Then the pair (*A*,*C*) of the system can be reduced to the form (5.9) by the use of the similarity transformation with the monomial matrix (5.25). Moreover the positive pair $(\hat{A}_1, \hat{C}_1)$ is observable.

Let

$$\hat{x}_i = Q x_i = \begin{bmatrix} \hat{x}_i^{(1)} \\ \hat{x}_i^{(2)} \end{bmatrix}, \quad \hat{x}_i^{(1)} \in \Re^{n_1}, \quad \hat{x}_i^{(2)} \in \Re^{n_2} \tag{5.28}$$

be a new state vector and

$$QB = \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{bmatrix}, \quad \hat{B}_1 \in \Re^{n_1 \times m}, \quad \hat{B}_2 \in \Re^{n_2 \times m} . \tag{5.29}$$

Definition 5.2. The positive subsystem

$$\hat{x}_i^{(1)} = \hat{A}_1 \hat{x}_i^{(1)} + \hat{B}_1 u_i \tag{5.30a}$$

$$y_i^{(1)} = \hat{C}_1 \hat{x}_i^{(1)} + D u_i \tag{5.30b}$$

is called the observable part of the system (2.1).

Theorem 5.3. The transfer matrix (4.41) of the positive system (2.1) is equal to the transfer matrix

$$T_1(z) = \hat{C}_1 [I_{n_1} z - \hat{A}_1]^{-1} \hat{B}_1 + D \tag{5.31}$$

of its observable part (5.30).
The proof is similar to the one of Theorem 4.3.
Therefore, the transfer matrix (5.31) represents only the observable part of the positive system (2.1)

6. CONCLUDING REMARKS

A new test for checking the reachability (observability) of positive discrete-time linear systems have been proposed (Theorem 3.1 and 3.2). Conditions have been established for: 1) decomposition of the pair (*A*,*B*) into reachable and unreachable parts (Theorem 4.1 and 4.2), 2) the decomposition of the pair (*A*,*C*) into observable and unobservable parts (Theorem 5.1 and 5.2).

It has been shown that the transfer matrix of the positive linear system is equal to the transfer matrix of its reachable (observable) part (Theorem 4.3 and 5.3). The considerations have been illustrated by numerical examples. Using the decomposition of the pair (A,B) and (A,C) it is possible to decompose a positive discrete-time linear system into four parts (subsystems): 1) reachable and observable part, 2) reachable and unobservable part 3) unreachable and observable part and 4) unreachable and unobservable part.
Open problems are extensions of these considerations to positive continuous-time linear systems and to positive 2D linear systems.

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