numerical simulation,
onset of instability,
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NUMERICAL SIMULATION OF NON-UNIFORM PROCESSES

In logistics as well as in other branches of science, engineering or transport, one frequently observes nonhomogeneous spatial or temporal patterns. It is a challenge for mathematical modeling and computer simulation to preserve such non-uniform behavior. Typically, the type of solutions to model equations depends on certain characteristic parameters. It is important to determine critical values of those.

SYMULACJA NUMERYCZNA PROCESÓW NIEJEDNORODNYCH

W logistyce, transporcie jak również innych dziedzinach naukowych, częstokroć występują zjawiska niejednorodne w czasie lub w przestrzeni. Poprawne odzwierciedlenie takiego zachowania stanowi wezwanie dla modelowania matematycznego i dla symulacji komputerowych. Typową okolicznością jest, że rozwiązania odpowiednich równań modelowych zależą od charakterystycznych parametrów. Istotne znacznenie ma obliczenie wartości krytycznych takich parametrów.

1. INTRODUCTION

Many processes in nature, engineering or economy can be easily described by means of differential equations, and often it is not very difficult to find certain special solutions to these *model equations*.

In particular, if intuition suggests e.g. symmetries or invariances of the studied phenomenon, numerical calculations of that type of solutions may be fast and effective. However, in many applications, solutions turn out to be non-unique, and in certain ranges of parameters non-trivial behavior has to be studied.

Let us consider some examples. We start with a dynamical system, described by an ordinary differential equation. The equations of motion of a rigid wheelset on a straight track under a constant driving moment admit a solution with vanishing lateral motion and constant speed along the middle line between the rails. However, if a certain *critical speed* is exceeded, the uniform solution becomes unstable and so-called *hunting* occurs, cf. [7, 2, 4].

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When there is a special extension of the model, we may have a breakdown of constant solutions in the space variable as well. Now we deal with partial differential equations. Again, the rolling of a railway wheel may serve as an example. Under ideal conditions, the wheel should roll at constant speed, and hence abrasive *wear* should occur all over its circumference at the same intensity. That way, the radius would shrink in a uniform way. At high speed this trivial solution becomes unstable, and patterns arise, which are called *polygonalisation*.

Similar phenomena are observed in various models. We studied in previous paper amongst others the loss of stability of compressed columns under follower load in [1], the breakdown of continuous families of equilibria in population dynamics in [5]. Wave-like behavior of solutions is obtained in models of traffic flow, which may be described either in a macroscopic way by hyperbolic balance laws (Lightwill-Whitman-Richards model) or in a microscopic way using a particle approach and/or cellular automata. Finally, the appearance of limit cycles in models with temporal delay has to be mentioned, cf. [6, 3].

2. COLLECTION OF MODELS

In this section we revisit three of the models mentioned in the Introduction, which may be of interest for logistics. First, there is a model describing the dynamics of three co-existing populations, inhabiting the same area. Relocation of one of the species may be a source or sink for the other ones, hence we have a coupling between state components. The same is true for the second example, poligonalization of wheels. This phenomenon has a major impact on high speed railway transportation and is still not fully understood. Finally, in a third example we briefly discuss the effect of retardation on model behavior. This is of particular interest, whenever a human factor is involved, e.g. the model of a driver in traffic simulation or a customer in market analysis.

2.1 Population dynamics

The evolution equation for the dynamics of three populations co-existing on a common territory is:

$$\dot{w}(t) = Kw'' + Mw' + F(w', w) := \Psi(w) \tag{1}$$

where: $w=(w_1,w_2,w_3)$ – population density vector,

K – matrix of diffusive mobility (diagonal),

M – matrix of advective couplings,

F – nonlinear interaction term.

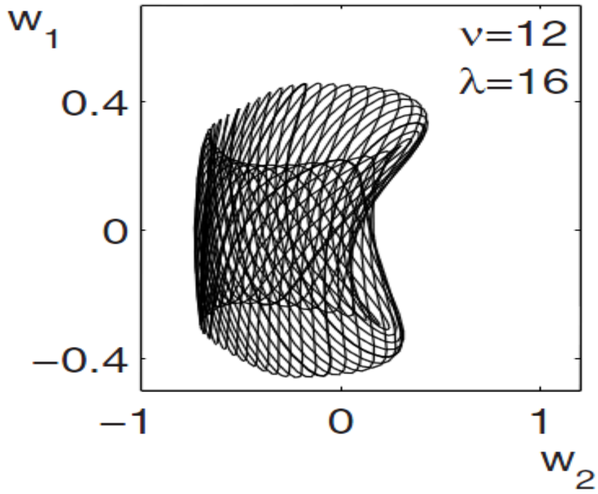

Fig.1. Orbit of means value in population dynamical model, projection on first two species

For a derivation we refer to [5]. Assuming positive diffusion coefficients, (1) is a coupled system of parabolic equations. Other than the well-known diffusion or heat transfer equations, due to the coupling, here non-trivial behavior can be observed. In dependence on the choice of parameters, in particular on the nonlinear term *F(w',w)*, non-unique stationary solutions may exist, but also oscillations and even chaotic trajectories have been found.

In Fig. 1, a torus-like orbit is shown. Most intriguingly, continuous families of stationary states may exist, see [5] and cited there papers. It has to be mentioned that a full survey of the parameter ranges is very complex. It is essential to fine-tune numerical algorithms to preserve the qualitative behavior of solutions in simulation results.

Fig. 2 shows a screen shot of a C++ program developed for the interactive scanning of the range of three model parameters. By manipulating the controls, interesting regimes are roughly localized, later on exact values of stationary configurations or limit cycles can be computed by suitable methods for solving nonlinear equations. Notice that the discretization of first-order derivatives in the nonlinear term *F(w',w)* is a very sensitive matter. Proper choices of parameters are selected by computer algebra methods, cf. [5].

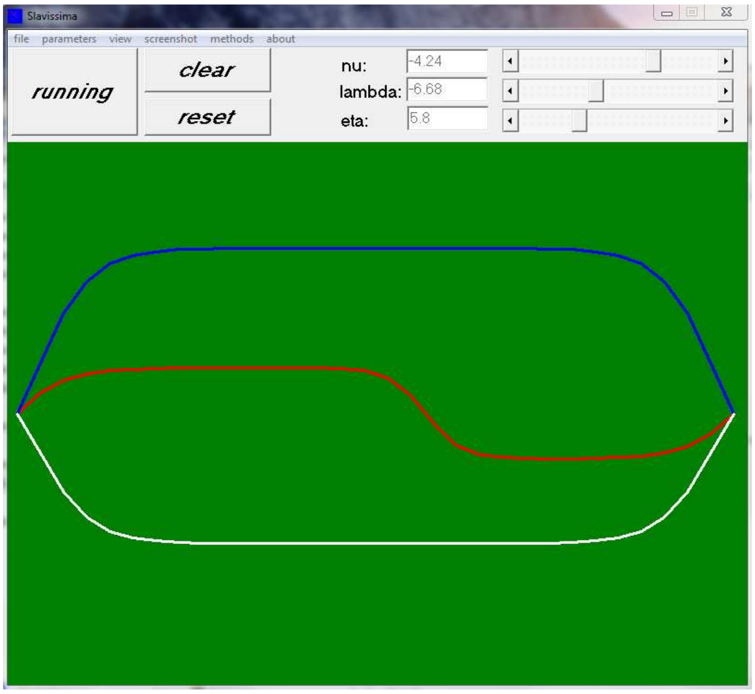

Fig.2. Screenshot of simulation tool for searching the 3d parameter space

2.2 Wheel surface evolution

The evolution of abrasive wear on the contact surface between rail and wheel is governed by two models, which are coupled in a complicated way into a feed-back loop.

On the one hand, the short-term dynamics of a rolling wheelset or a single rolling wheel is formulated in terms of a system of ordinary differential equations. This system expresses the balance of momentum, and it is parametrically dependent on the actual wheel geometry.

$$M\dot{z}(t) = -Kz(t) + f(t, z(t), r(.)) \tag{3}$$

where: z – state (position and velocity vectors),
t – time,
M – mass (matrix),
K – stiffness,
f – forces.

On the other hand, the wheel geometry is evolving slowly due to frictional effects in the contact zone. The speed of wear depends on the actual creepage, which in turn is determined by the dynamical wheel.

$$\dot{r}(t) = -\beta p(t, z(t), \dot{z}(t), r(.)) \tag{4}$$

where: *r* – the radius function (defined on [0, 2π]),
β – wear coefficient
p – frictional power calculated from (3)

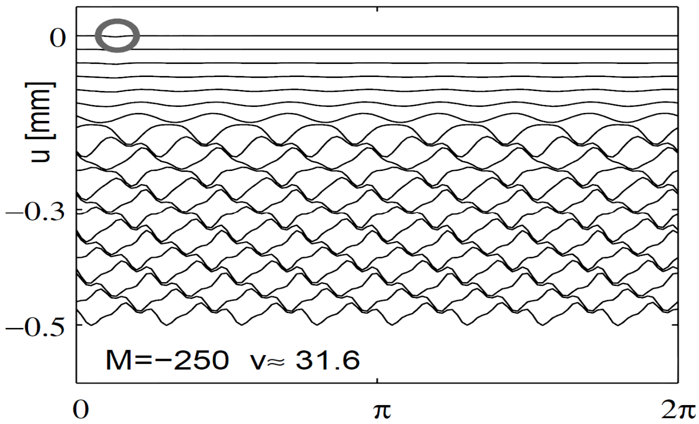

Fig.3. Evolution of wavy patterns on wheel surface

A chosen result of a computer simulation is shown in Fig. 3. Notice that poligonalization occurs only at certain ranges of the travelling speed. Careful speed control can even prevent pattern evolution or erase existing waves, cf. [4].

For details of the modeling, the proper choice of numerical methods and for more results we have to refer to [2, 4, 7].

2.3 Delay model

In many applications, system answers register with a certain delay, cf. [8]. Recently, a model problem of this type was studied analytically in [6], numerical calculations where reported in [3].

The evolution equation has the form

$$\dot{u}(t) = \big(a(u(t) - u(t - d(u(t)))\big) + g(u(t)) \tag{2}$$

where for example the nonlinear function *g* may have the form $g(u) = sign(u)u^2$.

Further, the delay function *d*, which is assumed constant in more trivial models, is here supposed to be positive and to decrease with increasing absolute value of the state variable *u*. This reflects human behavior – the farther the state deviates from normal, the more often we check for changes. The onset of instability for this class of models, described by (2), happens at a parameter value of *a=1.0*. For larger values, size and shape as well as the time period of limit cycles changes drastically, Fig. 4, right plot. The choice of the delay function *d* is essential, see Fig. 4, left part.

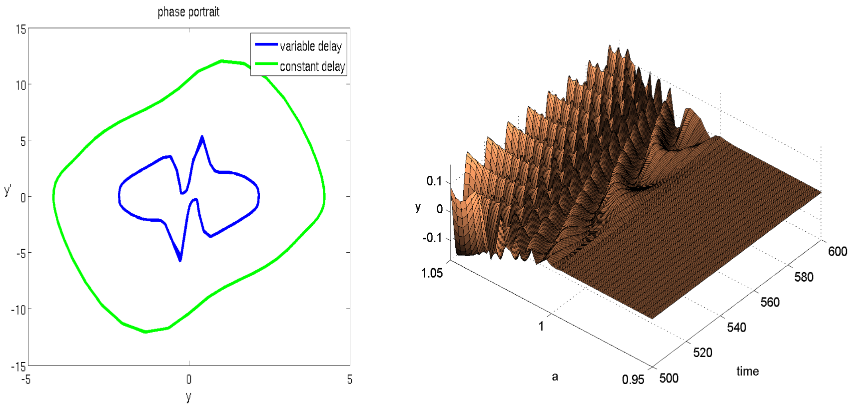

Fig.4. left: Limit cycles in delay model (green – constant delay, blue – variable delay)
Right: Onset of instability at a=1.0 and dependence of pattern on parameter

3. CONCLUSIONS

In logistics a wide range of model types is used to describe phenomena and processes. Variation of parameters may lead to qualitative changes in the behavior of solutions.

Best understood are models based on linear equations, in particular on ordinary differential equations. In that case a study of the spectrum of the evolution operator, i.e. the roots of the characteristic equation, cf. [1], gives accurate results. However, due to nonlinearity, advanced models rarely allow a full analytical solution, hence computer simulations are necessary.

The study of non-uniform solutions by computer simulation should always be verified by alternative algorithms and/or backed up by analytical considerations in order to distinguish between numerical artifacts and patterns of the true solution to the model equations, see [2, 4]. While a study of sensitivity to model parameters is typically a major goal, robustness with respect to parameters of numerical methods, such as step sizes, orders of polynomials, choice of grid types etc., must not be neglected.

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