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DECOMPOSITION OF DESCRIPTOR FRACTIONAL LINEAR SYSTEMS INTO DYNAMIC AND STATIC PARTS

A method for decomposition of descriptor fractional linear systems with regular pencils into dynamic and static parts is proposed. The method is based on modified version of the shuffle algorithm. A procedure of the decomposition is given and illustrated on numerical examples.

DEKOMPOZYCJA SINGULARNYCH LINOWYCH UKŁADÓW NIECAŁKOWITEGO RZĘDU NA CZĘŚĆ DYNAMICZNĄ I STATYCZNĄ

Zaproponowano metoda dekompozycji singularnych liniowych układów niecałkowitego rzędu o pęku regularnym na część dynamiczną oraz statyczną. Metoda oparta została na zmodyfikowanej wersji algorytmu przesuwania. Zaprezentowano procedurę dekompozycji która została zilustrowana przykładami numerycznymi.

1. INTRODUCTION

Descriptor (singular) linear systems have been addressed in many papers and books [1-4, 7, 8, 12]. The eigenvalues and invariants assignment by state and output feedbacks have been investigated in [1-3, 6, 12] and the realization problem for singular positive continuous-time systems with delays in [10]. The computation of Kronecker's canonical form of a singular pencil has been analyzed in [15].The fractional differential equations have been considered in the monograph [14]. Fractional positive linear systems have been addressed in [5, 9] and in the monograph [11]. Luenberger in [13] has proposed the shuffle algorithm to analysis of the singular linear systems.

In this paper a method for decomposition of the descriptor fractional linear systems with regular pencils into dynamic and static parts will be proposed. The method is based on the modified version of the shuffle algorithm.

The paper is organized as follows. In section 2 the decomposition method is presented for descriptor fractional discrete-time linear system. In section 3 the method is extended to the descriptor fractional continuous-time linear systems. Concluding remarks are given in section 4.

To the best of the author's knowledge the decomposition of descriptor fractional linear systems into dynamic and static parts has not been considered yet.

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The following notation will be used in the paper.

The set of $n \times m$ real matrices will denoted by $\Re^{n\times m}$ and $\Re^{n} := \Re^{n\times 1}$. The set of nonnegative integers will be denoted by Z_+ and the $n \times n$ identity matrix by I_n.

2. DESCRIPTOR FRACTIONAL DISCRETE-TIME LINEAR SYSTEMS

Consider the descriptor fractional discrete-time linear system

$$E\Delta^{\alpha} x_{i+1} = Ax_i + Bu_i, \; i \in Z_+ = \{0,1,...\} \tag{2.1}$$

where, $x_i \in \Re^n, u_i \in \Re^m$ are the state and input vectors, $A \in \Re^{n\times n}$, $E \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, and the fractional difference of the order α is defined by

$$\Delta^{\alpha} x_i = \sum_{k=0}^{i} (-1)^k \binom{\alpha}{k} x_{i-k}, \; 0 < \alpha < 1 \tag{2.2}$$

$$\binom{\alpha}{k} = \begin{cases} 1 & \text{for } k = 0 \\ \dfrac{\alpha(\alpha-1)...(\alpha-k+1)}{k!} & \text{for } k = 1,2,... \end{cases} \tag{2.3}$$

It is assumed that

$$\det E = 0 \tag{2.4a}$$

and the pencil is regular, i.e.

$$\det[Ez - A] \neq 0 \tag{2.4b}$$

for some $z \in C$ (the field of complex numbers).
Substituting (2.2) into (2.1) we obtain

$$\sum_{k=0}^{i+1} Ec_k x_{i-k+1} = Ax_i + Bu_i, \; i \in Z_+ \tag{2.5}$$

where

$$c_k = (-1)^k \binom{\alpha}{k} \tag{2.6}$$

The following elementary row operations will be used:

1) Multiplication of the ith row (column) by a real number c. This operation will be denoted by $L[i \times c]$ ($R[i \times c]$).
2) Addition to the ith row (column) of the jth row (column) multiplied by a real number c. This operation will be denoted by $L[i + j \times c]$ ($R[i + j \times c]$).
3) Interchange of the ith and jth rows (columns). This operation will be denoted by $L[i, j]$ ($R[i, j]$).

Applying the row elementary operations to (2.5) we obtain

$$\sum_{k=0}^{i+1}\begin{bmatrix}E_1\\0\end{bmatrix}c_k x_{i-k+1}=\begin{bmatrix}A_1\\A_2\end{bmatrix}x_i+\begin{bmatrix}B_1\\B_2\end{bmatrix}u_i,\ i\in Z_+ \tag{2.7}$$

where $E_1\in\Re^{n_1\times n}$ is full row rank and $A_1\in\Re^{n_1\times n}$, $A_2\in\Re^{(n-n_1)\times n}$, $B_1\in\Re^{n_1\times m}$, $B_2\in\Re^{(n-n_1)\times m}$. The equation (2.7) can be rewritten as

$$\sum_{k=0}^{i+1}E_1c_k x_{i-k+1}=A_1x_i+B_1u_i \tag{2.8a}$$

and

$$0=A_2x_i+B_2u_i \tag{2.8b}$$

Substituting in (2.8b) i by $i+1$ we obtain

$$A_2x_{i+1}=-B_2u_{i+1} \tag{2.9}$$

The equations (2.8a) and (2.9) can be written in the form

$$\begin{bmatrix}E_1\\A_2\end{bmatrix}x_{i+1}=\begin{bmatrix}A_1-c_1E_1\\0\end{bmatrix}x_i-\begin{bmatrix}c_2E_1\\0\end{bmatrix}x_{i-1}-...-\begin{bmatrix}c_{i+1}E_1\\0\end{bmatrix}x_0+\begin{bmatrix}B_1\\0\end{bmatrix}u_i+\begin{bmatrix}0\\-B_2\end{bmatrix}u_{i+1} \tag{2.10}$$

If the matrix

$$\begin{bmatrix}E_1\\A_2\end{bmatrix} \tag{2.11}$$

is singular then applying the row operations to (2.10) we obtain

$$\begin{bmatrix}E_2\\0\end{bmatrix}x_{i+1}=\begin{bmatrix}A_{20}\\\overline{A}_{20}\end{bmatrix}x_i+\begin{bmatrix}A_{21}\\\overline{A}_{21}\end{bmatrix}x_{i-1}+...+\begin{bmatrix}A_{2,i}\\\overline{A}_{2,i}\end{bmatrix}x_0+\begin{bmatrix}B_{20}\\\overline{B}_{20}\end{bmatrix}u_i+\begin{bmatrix}B_{21}\\\overline{B}_{21}\end{bmatrix}u_{i+1} \tag{2.12}$$

where $E_2\in\Re^{n_2\times n}$ is full row rank with $n_2\ge n_1$ and $A_{2,j}\in\Re^{n_2\times n}$, $\overline{A}_{2,j}\in\Re^{(n-n_2)\times n}$, $j=0,1,...,i$ $B_{2,k}\in\Re^{n_2\times m}$, $\overline{B}_{2,k}\in\Re^{(n-n_2)\times m}$, $k=0,1$.

Note that the array

$$\begin{matrix}E_1 & A_1-c_1E_1 & c_2E_1 & ... & c_{i+1}E_1 & B_1 & 0\\ A_2 & 0 & 0 & ... & 0 & 0 & -B_2\end{matrix} \tag{2.13}$$

corresponding to (2.10) can be obtained from

$$\begin{matrix} E_1 & A_1 - c_1E_1 & c_2E_1 & \dots & c_{i+1}E_1 & B_1 \\ 0 & A_2 & 0 & \dots & 0 & B_2 \end{matrix} \tag{2.14}$$

by the shuffle of A_2.
From (2.12) we have

$$0 = \overline{A}_{20}x_i + \overline{A}_{21}x_{i-1} + \dots + \overline{A}_{2,i}x_0 + \overline{B}_{20}u_i + \overline{B}_{21}u_{i+1} \tag{2.15}$$

Substituting in (2.15) i by $i + 1$ (in state vector x and in input u) we obtain

$$\overline{A}_{20}x_{i+1} = -\overline{A}_{21}x_i - \dots - \overline{A}_{2,i}x_1 - \overline{B}_{20}u_{i+1} - \overline{B}_{21}u_{i+2} \tag{2.16}$$

From (2.12) and (2.16) we have

$$\begin{bmatrix} E_2 \\ \overline{A}_{20} \end{bmatrix} x_{i+1} = \begin{bmatrix} A_{20} \\ -\overline{A}_{21} \end{bmatrix} x_i + \begin{bmatrix} A_{21} \\ -\overline{A}_{22} \end{bmatrix} x_{i-1} + \dots + \begin{bmatrix} A_{2,i} \\ 0 \end{bmatrix} x_0 + \begin{bmatrix} B_{20} \\ 0 \end{bmatrix} u_i + \begin{bmatrix} B_{21} \\ -\overline{B}_{20} \end{bmatrix} u_{i+1} + \begin{bmatrix} 0 \\ -\overline{B}_{21} \end{bmatrix} u_{i+2} \tag{2.17}$$

If the matrix

$$\begin{bmatrix} E_2 \\ \overline{A}_{20} \end{bmatrix} \tag{2.18}$$

is singular then we repeat the procedure. Continuing this procedure after finite number of steps p we obtain

$$\begin{bmatrix} E_p \\ 0 \end{bmatrix} x_{i+1} = \begin{bmatrix} A_{p,0} \\ \overline{A}_{p,0} \end{bmatrix} x_i + \begin{bmatrix} A_{p,1} \\ \overline{A}_{p,2} \end{bmatrix} x_{i-1} + \dots + \begin{bmatrix} A_{pi} \\ \overline{A}_{pi} \end{bmatrix} x_0 + \begin{bmatrix} B_{p,0} \\ \overline{B}_{p,0} \end{bmatrix} u_i + \begin{bmatrix} B_{p,1} \\ \overline{B}_{p,1} \end{bmatrix} u_{i+1} + \dots + \begin{bmatrix} B_{p,p-1} \\ \overline{B}_{p,p-1} \end{bmatrix} u_{i+p-1} \tag{2.19}$$

where $E_p \in \Re^{n_p \times n}$ is full row rank, $A_{pj} \in \Re^{n_p \times n}$, $\overline{A}_{pj} \in \Re^{(n-n_p)\times n}$, $j = 0,1,\dots, p$ and $B_{pk} \in \Re^{n_p \times m}$, $\overline{B}_{pk} \in \Re^{(n-n_p)\times m}$, $k = 0,1,\dots, p-1$ with nonsingular matrix

$$\begin{bmatrix} E_p \\ \overline{A}_{p,0} \end{bmatrix} \in \Re^{n \times n} \tag{2.20}$$

Using the elementary column operations we may reduce the matrix (2.20) to the form

$$\begin{bmatrix} I_{n_p} & 0 \\ A_{21} & I_{n-n_p} \end{bmatrix}, \quad A_{21} \in \Re^{(n-n_p)\times n_p}. \tag{2.21}$$

Performing the same elementary operations on the matrix I_n we can find the matrix $Q \in \Re^{n\times n}$ such that

$$\begin{bmatrix} E_p \\ \overline{A}_{p,0} \end{bmatrix} Q = \begin{bmatrix} I_{n_p} & 0 \\ A_{21} & I_{n-n_p} \end{bmatrix}. \tag{2.22}$$

Taking into account (2.22) and defining the new state vector

$$\tilde{x}_i = Q^{-1} x_i = \begin{bmatrix} \tilde{x}_i^{(1)} \\ \tilde{x}_i^{(2)} \end{bmatrix}, \quad \tilde{x}_i^{(1)} \in \Re^{n_p}, \quad \tilde{x}_i^{(2)} \in \Re^{n-n_p}, \quad i \in Z_+ \tag{2.23}$$

from (2.19) we obtain

$$\begin{aligned} \tilde{x}_{i+1}^{(1)} &= E_p x_{i+1} = E_p Q Q^{-1} x_{i+1} = A_{p,0} Q Q^{-1} x_i + A_{p,1} Q Q^{-1} x_{i-1} + \ldots + A_{pi} Q Q^{-1} x_0 \\ &\quad + B_{p,0} u_i + B_{p,1} u_{i+1} + \ldots + B_{p,p-1} u_{i+p-1} \\ &= [A_{p,0}^{(1)} \quad A_{p,0}^{(2)}] \begin{bmatrix} \tilde{x}_i^{(1)} \\ \tilde{x}_i^{(2)} \end{bmatrix} + [A_{p,1}^{(1)} \quad A_{p,1}^{(2)}] \begin{bmatrix} \tilde{x}_{i-1}^{(1)} \\ \tilde{x}_{i-1}^{(2)} \end{bmatrix} + \ldots + [A_{pi}^{(1)} \quad A_{pi}^{(2)}] \begin{bmatrix} \tilde{x}_0^{(1)} \\ \tilde{x}_0^{(2)} \end{bmatrix} \\ &\quad + B_{p,0} u_i + B_{p,1} u_{i+1} + \ldots + B_{p,p-1} u_{i+p-1} \\ &= A_{p,0}^{(1)} \tilde{x}_i^{(1)} + A_{p,0}^{(2)} \tilde{x}_i^{(2)} + \ldots + A_{pi}^{(1)} \tilde{x}_0^{(1)} + A_{pi}^{(2)} \tilde{x}_0^{(2)} \\ &\quad + B_{p,0} u_i + B_{p,1} u_{i+1} + \ldots + B_{p,p-1} u_{i+p-1}, \quad i \in Z_+ \end{aligned} \tag{2.24}$$

and

$$\begin{aligned} \tilde{x}_i^{(2)} &= -A_{21} \tilde{x}_i^{(1)} - \overline{A}_{p,1}^{(1)} \tilde{x}_{i-1}^{(1)} - \overline{A}_{p,1}^{(2)} \tilde{x}_{i-1}^{(2)} - \ldots - \overline{A}_{pi}^{(1)} \tilde{x}_0^{(1)} - \overline{A}_{pi}^{(2)} \tilde{x}_0^{(2)} \\ &\quad - \overline{B}_{p,0} u_i - \ldots - \overline{B}_{p,p-1} u_{i+p-1}, \quad i \in Z_+ \end{aligned} \tag{2.25}$$

where

$$A_{pj} Q = [A_{pj}^{(1)} \quad A_{pj}^{(2)}], \quad \overline{A}_{pj} = [\overline{A}_{pj}^{(1)} \quad \overline{A}_{pj}^{(2)}], \quad j = 0,1,\ldots,i \tag{2.26}$$

Substitution of (2.25) into (2.24) yields

$$\tilde{x}_{i+1}^{(1)} = \tilde{A}_{p,0} \tilde{x}_i^{(1)} + \ldots + \tilde{A}_{pi} \tilde{x}_0^{(1)} + \tilde{B}_{p,0} u_i + \ldots + \tilde{B}_{p,p-1} u_{i+p-1}, \quad i \in Z_+ \tag{2.27}$$

where

$$\begin{aligned} \tilde{A}_{p,0} &= A_{p,0}^{(1)} - A_{p,0}^{(2)} A_{21}, \ldots, \tilde{A}_{pi} = A_{pi}^{(1)} - A_{p,0}^{(2)} \overline{A}_{pi}^{(1)} \\ \tilde{B}_{p,0} &= B_{p,0} - A_{p,0}^{(2)} \overline{B}_{p,0}, \ldots, \tilde{B}_{p,p-1} = B_{p,p-1} - A_{p,0}^{(2)} \overline{B}_{p,p-1} \end{aligned} \tag{2.28}$$

The standard system described by the equation (2.27) is called the dynamic part of the system (2.5) and the system described by the equation (2.25) is called the static part of the system (2.5).

The procedure can be justified as follows. The elementary row operations do not change the rank of the matrix $[Ez - A]$. The substitution in the equations (2.8b) and (2.15) i by $i + 1$ also does not change the rank of the matrix $[Ez - A]$ since it is equivalent to multiplication of its lower rows by z and by assumption (2.4b) holds. Therefore, the following theorem has been proved.

Theorem 1. The descriptor fractional discrete-time linear system (2.5) satisfying the assumption (2.4) can be decomposed into the dynamic part (2.27) and static part (2.25).

Example 1. Consider the descriptor fractional linear system (2.1) for $\alpha = 0.5$ with

$$E = \begin{bmatrix} 5 & 0 & 2 \\ 2 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0.2 & 2 & -2 \\ 2 & 1 & 0 \\ -1.8 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ -1 & 2 \\ 2 & -1 \end{bmatrix} \tag{2.29}$$

In this case the conditions (2.4) are satisfied since

$$\det E = 0 \text{ and } \det[Ez - A] = \begin{vmatrix} 5z - 0.2 & -2 & 2z + 2 \\ 2z - 2 & -1 & z \\ z + 1.8 & 0 & 1 \end{vmatrix} = z - 0.2$$

Applying to the matrices (2.29) the following elementary row operations $L[1 + 2 \times (-2)]$, $L[3 + 1 \times (-1)]$ we obtain

$$[E \quad A \quad B] = \begin{bmatrix} 5 & 0 & 2 & 0.2 & 2 & -2 & 1 & 2 \\ 2 & 0 & 1 & 2 & 1 & 0 & -1 & 2 \\ 1 & 0 & 0 & -1.8 & 0 & -1 & 2 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -3.8 & 0 & -2 & 3 & -2 \\ 2 & 0 & 1 & 2 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 2 & 0 & 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} E_1 & A_1 & B_1 \\ 0 & A_2 & B_2 \end{bmatrix} \tag{2.30}$$

and the equations (2.8) have the form

$$\sum_{k=0}^{i+1} c_k \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix} x_{i-k+1} = \begin{bmatrix} -3.8 & 0 & -2 \\ 2 & 1 & 0 \end{bmatrix} x_i + \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix} u_i \tag{2.31a}$$

and

$$0 = [2 \quad 0 \quad 1] x_i + [-1 \quad 1] u_i \tag{2.31b}$$

Using (2.6) we obtain $c_1 = -\binom{\alpha}{1} = -\alpha = -0.5$, $c_2 = (-1)^2 \binom{\alpha}{2} = \frac{\alpha(\alpha-1)}{2!} = -\frac{1}{8}$, …, $c_{i+1} = (-1)^{i-1} \left. \frac{\alpha(\alpha-1)...(\alpha-i)}{(i+1)!} \right|_{\alpha=0.5}$ and the equation (2.10) has the form

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 2 & 0 & 1 \end{bmatrix} x_{i+1} = \begin{bmatrix} -3.3 & 0 & -2 \\ 3 & 1 & 0.5 \\ 0 & 0 & 0 \end{bmatrix} x_i + \frac{1}{8} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x_{i-1} \\ - ... - c_{i+1} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x_0 + \begin{bmatrix} 3 & -2 \\ -1 & 2 \\ 0 & 0 \end{bmatrix} u_i + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & -1 \end{bmatrix} u_{i+1} \tag{2.32}$$

The matrix $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 2 & 0 & 1 \end{bmatrix}$ is singular. Performing the elementary row operation $L[3 + 2 \times (-1)]$ on (2.32) we obtain the following

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x_{i+1} = \begin{bmatrix} -3.3 & 0 & -2 \\ 3 & 1 & 0.5 \\ -3 & -1 & -0.5 \end{bmatrix} x_i + \frac{1}{8} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ -2 & 0 & -1 \end{bmatrix} x_{i-1} \\ - ... - c_{i+1} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ -2 & 0 & -1 \end{bmatrix} x_0 + \begin{bmatrix} 3 & -2 \\ -1 & 2 \\ 1 & -2 \end{bmatrix} u_i + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & -1 \end{bmatrix} u_{i+1} \tag{2.33}$$

The matrix

$$\begin{bmatrix} E_2 \\ \overline{A}_{20} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ -3 & -1 & -0.5 \end{bmatrix} \tag{2.34}$$

is nonsingular and to reduce this matrix to the form (3.21) we perform the elementary column operations $R[1 + 3 \times (-2)]$, $R[2 \times (-1)]$, $R[2,3]$. The matrix Q has the form

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ -2 & 1 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} E_2 \\ \overline{A}_{20} \end{bmatrix} Q = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ -3 & -1 & -0.5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ -2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & -0.5 & 1 \end{bmatrix}, \quad A_{21} = [-2 \quad -0.5], \quad n_2 = 2$$

The new state vector (2.23) is

$$\tilde{x}_i = Q^{-1} x_i = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_{1,i} \\ x_{2,i} \\ x_{3,i} \end{bmatrix} = \begin{bmatrix} \tilde{x}_i^{(1)} \\ \tilde{x}_i^{(2)} \end{bmatrix}, \quad \tilde{x}_i^{(1)} = \begin{bmatrix} x_{1,i} \\ 2x_{1,i} + x_{3,i} \end{bmatrix}, \quad \tilde{x}_i^{(2)} = -x_{2,i} \tag{2.35}$$

In this case the equations (2.24) and (2.25) have the forms

$$\tilde{x}_{i+1}^{(1)} = \begin{bmatrix} 0.7 & -2 \\ 2 & 0.5 \end{bmatrix} \tilde{x}_i^{(1)} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \tilde{x}_i^{(2)} + \frac{1}{8} \tilde{x}_{i-1}^{(1)} - \ldots - c_{i+1} \tilde{x}_0^{(1)} + \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix} u_i \tag{2.36}$$

and

$$\tilde{x}_i^{(2)} = [2 \quad 0.5] \tilde{x}_i^{(1)} + [0.25 \quad 0] \tilde{x}_{i-1}^{(1)} + \ldots + c_{i+1} [-2 \quad 0] \tilde{x}_0^{(1)} - [1 \quad -2] u_i - [1 \quad -1] u_{i+1} \tag{2.37}$$

Substituting (2.37) into (2.36) we obtain

$$\tilde{x}_{i+1}^{(1)} = \begin{bmatrix} 0.7 & -2 \\ 0 & 0 \end{bmatrix} \tilde{x}_i^{(1)} + \frac{1}{8} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tilde{x}_{i-1}^{(1)} - \ldots - c_{i+1} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \tilde{x}_0^{(1)} + \begin{bmatrix} 3 & -2 \\ 0 & 0 \end{bmatrix} u_i + \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} u_{i+1} \tag{2.38}$$

The dynamic part of the system is described by (2.38) and the static part by (2.37).

3. DESCRIPTOR FRACTIONAL CONTINUOUS-TIME LINEAR SYSTEMS

The following Caputo definition of the fractional derivative will be used [11]

$$D^\alpha f(t) = \frac{d^\alpha}{dt^\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau, \quad n-1 < \alpha \le n \in N = \{1,2,\ldots\} \tag{3.1}$$

where $\alpha \in \Re_+$ is the order of fractional derivative and $f^{(n)}(\tau) = \frac{d^n f(\tau)}{d\tau^n}$ and $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$ is a gamma function.

Consider the descriptor fractional continuous-time linear system described by the state equation

$$E \frac{d^\alpha}{dt^\alpha} x(t) = Ax(t) + Bu(t), \quad 0 < \alpha \le 1 \tag{3.2}$$

where $x(t) \in \Re^n$, $u(t) \in \Re^m$ are the state and input vectors and $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$.
It is assumed that $\det E = 0$ and

$$\det[Es - A] \neq 0 \text{ for some } s \in C. \tag{3.3}$$

Performing elementary row operations on the array

$$\begin{matrix} E & A & B \end{matrix}$$

(or equivalently on the equation (3.2)) we obtain

$$\begin{matrix} E_1 & A_1 & B_1 \\ 0 & A_2 & B_2 \end{matrix} \tag{3.4}$$

and

$$E_1 \frac{d^\alpha}{dt^\alpha} x(t) = A_1 x(t) + B_1 u(t) \tag{3.5a}$$

$$0 = A_2 x(t) + B_2 u(t) \tag{3.5b}$$

where $E_1 \in \Re^{r_1\times n}$ has full row rank. Differentiation of (3.5b) with respect to time yields

$$A_2 \frac{d^\alpha}{dt^\alpha} x(t) = -B_2 \frac{d^\alpha}{dt^\alpha} u(t) \tag{3.6}$$

The equations (3.5a) and (3.6) can be written in the form

$$\begin{bmatrix} E_1 \\ A_2 \end{bmatrix} \frac{d^\alpha}{dt^\alpha} x(t) = \begin{bmatrix} A_1 \\ 0 \end{bmatrix} x(t) + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ -B_2 \end{bmatrix} \frac{d^\alpha}{dt^\alpha} u(t) \tag{3.7}$$

The array

$$\begin{matrix} E_1 & A_1 & B_1 & 0 \\ A_2 & 0 & 0 & -B_2 \end{matrix} \tag{3.8}$$

(or equivalently the equation (3.7)) can be obtained from (3.4) by the shuffle of A_2.
If matrix $\begin{bmatrix} E_1 \\ A_2 \end{bmatrix}$ is singular then we repeat the step of the procedure for (3.7) and after finite numbers of steps (in a similar way as for discrete-time systems) we obtain

$$\begin{bmatrix} E_p \\ 0 \end{bmatrix} \frac{d^\alpha}{dt^\alpha} x(t) = \begin{bmatrix} A_p \\ \overline{A}_p \end{bmatrix} x(t) + \begin{bmatrix} B_{p0} \\ \overline{B}_{p0} \end{bmatrix} u(t) + \begin{bmatrix} B_{p1} \\ \overline{B}_{p1} \end{bmatrix} \frac{d^\alpha}{dt^\alpha} u(t) + ... + \begin{bmatrix} B_{p,p-1} \\ \overline{B}_{p,p-1} \end{bmatrix} \frac{d^{(p-1)\alpha}}{dt^{(p-1)\alpha}} u(t) \quad (3.9)$$

where $E_p \in \Re^{r_p \times n}$ has full row rank and the matrix

$$\begin{bmatrix} E_p \\ \overline{A}_p \end{bmatrix} \quad (3.10)$$

is nonsingular.
Using the elementary column operations we may reduce the matrix (3.10) to the form

$$\begin{bmatrix} I_{n_p} & 0 \\ A_{21} & I_{n-n_p} \end{bmatrix} \in \Re^{(n-n_p) \times n_p} \quad (3.11)$$

and find the matrix $Q \in \Re^{n \times n}$ such that

$$\begin{bmatrix} E_p \\ \overline{A}_p \end{bmatrix} Q = \begin{bmatrix} I_{n_p} & 0 \\ A_{21} & I_{n-n_p} \end{bmatrix}. \quad (3.12)$$

Defining the new state vector

$$\overline{x}(t) = Q^{-1} x(t) = \begin{bmatrix} \overline{x}_1(t) \\ \overline{x}_2(t) \end{bmatrix}, \ \overline{x}_1(t) \in \Re^{n_p}, \ \overline{x}_2(t) \in \Re^{n-n_p} \quad (3.13)$$

from (3.9) we obtain

$$\begin{aligned} \frac{d^\alpha}{dt^\alpha} \overline{x}_1(t) &= A_p Q Q^{-1} x(t) + B_{p0} u(t) + B_{p1} \frac{d^\alpha}{dt^\alpha} u(t) + ... + B_{p,p-1} \frac{d^{(p-1)\alpha}}{dt^{(p-1)\alpha}} u(t) \\ &= [A_{p1} \quad A_{p2}] \begin{bmatrix} \overline{x}_1(t) \\ \overline{x}_2(t) \end{bmatrix} + B_{p0} u(t) + B_{p1} \frac{d^\alpha}{dt^\alpha} u(t) + ... + B_{p,p-1} \frac{d^{(p-1)\alpha}}{dt^{(p-1)\alpha}} u(t) \end{aligned} \quad (3.14a)$$

and

$$\overline{x}_2(t) = -\overline{A}_{21} \overline{x}_1(t) - \overline{B}_{p0} u(t) - \overline{B}_{p1} \frac{d^\alpha}{dt^\alpha} u(t) - ... - \overline{B}_{p,p-1} \frac{d^{(p-1)\alpha}}{dt^{(p-1)\alpha}} u(t) \quad (3.14b)$$

where

$$[A_{p1} \quad A_{p2}] = A_p Q, \ \ A_{p1} \in \Re^{n_p \times n_p}, \ \ A_{p2} \in \Re^{n_p \times (n-n_p)}.$$

Substitution of (3.14b) into (3.14a) yields

$$\frac{d^{\alpha}}{dt^{\alpha}}\bar{x}_1(t) = \bar{A}_1\bar{x}_1(t) + \bar{B}_0 u(t) + \bar{B}_1\frac{d^{\alpha}}{dt^{\alpha}}u(t) + ... + \bar{B}_{p-1}\frac{d^{(p-1)\alpha}}{dt^{(p-1)\alpha}}u(t) \tag{3.15a}$$

where

$$\bar{A}_1 = A_{p1} - A_{p2}\bar{A}_{21},\ \ \bar{B}_0 = B_{p0} - A_{p2}\bar{B}_{p0},\ \ \bar{B}_1 = B_{p1} - A_{p2}\bar{B}_{p1},...,\bar{B}_{p-1} = B_{p,p-1} - A_{p2}\bar{B}_{p,p-1}. \tag{3.15b}$$

The standard system described by the equation (3.15a) is called the dynamic part of the system (3.2) and the system described by the equation (3.14b) is called the static part of the system (3.2). The procedure can be justified in a similar way as for the discrete-time systems.

Therefore, the following theorem has been proved.

Theorem 2. The descriptor fractional continuous-time linear system (3.2) satisfying the assumption (3.3) can be decomposed into the dynamic part (3.15a) and the static part (3.14b).

Example 2. Consider the descriptor fractional linear system (3.2) with matrices

$$E = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ \ A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ \ B = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \tag{3.16}$$

The matrices (3.16) satisfy the condition (3.3) since

$$\det[Es - A] = \begin{vmatrix} s & -1 & s \\ -1 & s & 0 \\ 0 & 0 & -1 \end{vmatrix} = -s^2 + 1 \tag{3.17}$$

From (3.16) we have

$$E_1\frac{d^{\alpha}}{dt^{\alpha}}x(t) = A_1 x(t) + B_1 u(t) \tag{3.18a}$$

$$0 = A_2 x(t) + B_2 u(t) \tag{3.18b}$$

where

$$E_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},\ \ A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},\ \ B_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix},\ \ A_2 = [0 \quad 0 \quad 1],\ \ B_2 = [2].$$

Differentiation with respect to time of (3.18b) yields

$$A_2\frac{d^{\alpha}}{dt^{\alpha}}x(t) = -B_2\frac{d^{\alpha}}{dt^{\alpha}}u(t) \tag{3.19}$$

The equations (3.18a) and (3.19) can be written in the form

$$\begin{bmatrix} E_1 \\ A_2 \end{bmatrix} \frac{d^\alpha}{dt^\alpha} x(t) = \begin{bmatrix} A_1 \\ 0 \end{bmatrix} x(t) + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ -B_2 \end{bmatrix} \frac{d^\alpha}{dt^\alpha} u(t) \tag{3.20}$$

The matrix

$$\begin{bmatrix} E_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.21}$$

is nonsingular. Performing the elementary column operation $R[3+1\times(-1)]$ on (3.21) we obtain the identity matrix I_3 and

$$Q = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.22}$$

such that

$$\begin{bmatrix} E_1 \\ A_2 \end{bmatrix} Q = I_3 . \tag{3.23}$$

Defining the new state vector

$$\bar{x}(t) = Q^{-1} x(t) = \begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \end{bmatrix}, \ \bar{x}_1(t) = \begin{bmatrix} x_1(t) + x_3(t) \\ x_2(t) \end{bmatrix}, \ \bar{x}_2(t) = x_3(t) \tag{3.24}$$

from (3.20) we obtain

$$\frac{d^\alpha}{dt^\alpha} \bar{x}_1(t) = E_1 Q \bar{x}(t) = A_1 \bar{x}(t) + B_1 u(t) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \bar{x}(t) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u(t) \tag{3.25a}$$

$$\bar{x}_2(t) = x_3(t) = -2u(t) \tag{3.25b}$$

The dynamic part of the system is described by the equation (3.25a) and the static part by the equation (3.25b).

4. CONCLUDING REMARKS

A method for decomposition of descriptor fractional discrete-time and continuous-time linear systems with regular pencils into dynamic and static parts has been proposed. The method is based on modified version of the shuffle algorithm. It has been shown that descriptor linear system can be decomposed if their pencils are regular (Theorem 1 and 2) The procedure of the decomposition has been demonstrated on numerical examples. Open problems are extension of these considerations to positive descriptor fractional linear systems and to descriptor fractional linear systems with singular pencils.

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